Lecture 11 (c) ENEE664 Spring 2004

Second Order Necessary Conditions in the Calculus of Variations

Consider the functional

$$J[x] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\, dt$$

defined on $D[t_1, t_2]$ for L sufficiently differentiable.

Suppose $x(t_1) = x_1$ and $x(t_2) = x_2$ are fixed. Define the variation of J

$$J^{\varepsilon}[x] = \int_{t_1}^{t_2} L(t, x(t) + \varepsilon h(t), \dot{x}(t) + \varepsilon \dot{h}(t))\, dt$$

under a variation of x to $x + \varepsilon h$ where $h(t_1) = h(t_2) = 0$.

$$\left.\frac{dJ^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t))\, h(t) + \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\, \dot{h}(t) \right) dt$$

and

$$\left.\frac{d^2 J^{\varepsilon}}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_{t_1}^{t_2} \left[\frac{\partial^2 L}{\partial x^2}(t, x(t), \dot{x}(t))\, h^2(t) + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}}(t, x(t), \dot{x}(t))\, h(t) \dot{h}(t) + \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}}(t, x(t), \dot{x}(t))\, \dot{h}^2(t) \right] dt$$

We would like to refer to $\left.\frac{d^2 J^\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}$

above as the <u>second variation</u> denoted by

$$\delta^2 J[h] = \int_{t_1}^{t_2} \left[\frac{\partial^2 L}{\partial \dot{x}\, \partial \dot{x}} \cdot \dot{h}^2 + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} \cdot h\dot{h} + \frac{\partial^2 L}{\partial x^2} h^2 \right] dt$$

We can get rid of the $h\dot{h}$ term in the integral by integration by parts (we have assumed sufficient differentiability). Observe

$$\int_{t_1}^{t_2} 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} h\dot{h}\, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{d}{dt}(h^2)\, dt$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) h^2(t)\, dt$$

$$+ \left.\frac{\partial^2 L}{\partial x\, \partial \dot{x}} h^2 \right|_{t_1}^{t_2}$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) h^2(t)\, dt.$$

Thus $\delta^2 J[h] = \int_{t_1}^{t_2} \left[P(t)\, \dot{h}^2(t) + Q(t)\, h^2(t) \right] dt$, where,

$$P(t) = \frac{\partial^2 L}{\partial \dot{x}\, \partial \dot{x}}(t, x(t), \dot{x}(t)) \; ; \quad Q(t) = \frac{\partial^2 L}{\partial x^2}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right)$$

Theorem (second order necessary conditions).

Suppose $J[x]$ attains a local minimum at x ($\Rightarrow$ $J^{\varepsilon}[x]$ attains a local minimum at $\varepsilon = 0$). Then :

(a) $\left.\dfrac{dJ^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = 0 \quad \leftrightarrow \quad$ E-L holds

(b) $\left.\dfrac{d^2 J^{\varepsilon}}{d\varepsilon^2}\right|_{\varepsilon=0} \geq 0 \quad \Rightarrow$

(Legendre) $\qquad \dfrac{\partial^2 L}{\partial \dot{x} \partial \dot{x}}(t, x(t), \dot{x}(t)) \geq 0$

at each t.

Proof (a) is already known

(b) That $\dfrac{d^2 J^{\varepsilon}}{d\varepsilon^2} \geq 0$ is a special case of Theorem on necessary conditions in Lecture 11 (a) page 1.

The Legendre conditions follows from this.

To see this suppose not : say

$$P(t) \triangleq \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}}(t_0, x(t_0), \dot{x}(t_0)) = -2\beta, \qquad \beta > 0$$

for some $t_0 \in [t_1, t_2]$.

By continuity of $P(t)$, there exits an $\alpha > 0$ such that ~~...~~ $t_1 \le t_0 - \alpha$, $t_0 + \alpha \le t_2$, and

$$P(t) < -\beta \qquad t_0 - \alpha \le t \le t_0 + \alpha$$

Consider $\tilde{h}(t) \in D_3(t_1, t_2)$ such that

$$\tilde{h}(t) = \begin{cases} \sin^2 \dfrac{\pi(t-t_0)}{\alpha} & t_0 - \alpha \le t \le t_0 + \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then $$\int_{t_1}^{t_2} \left(P(t)\,\dot{\tilde{h}}(t)^2 + Q(t)\,\tilde{h}^2(t)\right) dt$$

$$= \int_{t_0-\alpha}^{t_0+\alpha} P(t) \frac{\pi^2}{\alpha^2} \sin^2\left(\frac{2\pi}{\alpha}(t-t_0)\right) dt + \int_{t_0-\alpha}^{t_0+\alpha} Q(t) \sin^4 \frac{\pi(t-t_0)}{\alpha}\, dt$$

$$< -\beta \frac{\pi^2}{\alpha^2} 2\alpha + 2M\alpha .$$

(where $M = \max\limits_{[t_1, t_2]} |Q(t)|$)

for sufficiently small α the r.h.s. above becomes negative which is a contradiction.

Thus $P(t) \ge 0 \quad t \in [t_1, t_2]$ (Legendre) is necessary ▨

Legendre unsuccessfully attempted to turn this into a sufficient condition by the strengthened condition

$$P(t) > 0$$

(analogous to argument in Lecture 11(a) page 2), and the following sophisticated completion of squares trick.

Observe, for an arbitrary differentiable $w(t)$,

$$0 = \int_{t_1}^{t_2} \frac{d}{dt}\left(w(t)\, h^2(t)\right) dt$$

$$= \dot{w}\, h^2(t) + 2w\, h(t)\dot{h}(t)$$

Adding this to $\delta^2 J[h]$ we get

$$\delta^2 J[h] = \int_{t_1}^{t_2} \left(P(t)\, \dot{h}^2(t) + Q(t)\, h^2(t)\right) dt$$

$$= \int_{t_1}^{t_2} \Big[P(t)\, \dot{h}^2(t) + 2w(t)h(t)\dot{h}(t) + \left(Q(t) + \dot{w}(t)\right) h^2(t)\Big]\, dt$$

$$= \int_{t_1}^{t_2} \left[P^{1/2}(t)\, \dot{h}(t) + (Q(t) + \dot{w}(t))^{1/2}\, h(t) \right]^2 dt$$

$$P(t)\,(Q(t) + \dot{w}(t)) = w^2(t)$$

If w could be found then $\delta^2 J[h] > 0$ $\forall\, h \not\equiv 0$, and sufficiency applies.

The catch is that the Riccati equation

$$P(t)\,(Q(t) + \dot{w}(t)) = w^2(t)$$

need not have a solution $w(t)$ ~~for~~ over the entire interval $[t_1, t_2]$. Finite escape time for Riccati equations creates a problem. We need

<u>Defn</u> (Conjugate Points).

A point $\bar{a} \neq t_1$, $t_1 < \bar{a} \leq t_2$ is said to ~~be~~ be conjugate to t_1 if the equation,

$$\frac{d}{dt}\left(P(t)\,\dot{h}(t)\right) = Q(t)\,h(t)$$

has a solution which vanishes $t = t_1$ and $t = \bar{a}$ but is <u>not identically</u> zero on $[t_1, \bar{a}]$ ☒

<u>Remark</u>. The above differential equation is simply the Euler Lagrange equation for the quadratic functional

$$\int_{t_1}^{t_2} \left[P(t)\,\dot{h}^2(t) + Q(t)\,h^2(t)\right] dt .$$

<u>Theorem</u>

If $P(t) > 0$ on $[t_1, t_2]$ and if the interval $[t_1, t_2]$ contains <u>no points conjugate</u> to t_1, then the quadratic functional

$$\int_{t_1}^{t_2} \left[P(t)\,\dot{h}^2(t) + Q(t)\,h^2(t)\right] dt$$

is positive definite for all $h(t)$ s.t. $h(t_1) = h(t_2) = 0$.

Proof Consider the equation

$$\frac{d}{dt}\left(P(t)\,\dot u(t)\right) = Q(t)\,u(t).$$

In the absence of conjugate points to t_1 (by hypothesis), this equation has a solution $u(t)$ which does not vanish anywhere on the interval $[t_1, t_2]$

Let $$w(t) = -\frac{\dot u(t)}{u(t)}\,P(t)$$

Then $$\dot w = \frac{1}{u^2}\,\dot u\,\dot u\,P(t) - \frac{1}{u}\frac{d}{dt}\left(\dot u(t)P(t)\right) =$$

Then $$P(t)\left(Q(t) + \dot w(t)\right)$$

$$= P(t)\left(\frac{1}{u^2}\,\dot u^2 P(t) - \frac{1}{u}\frac{d}{dt}\left(\dot u(t)P(t)\right) + Q(t)\right)$$

$$= \frac{\dot u^2(t)\,P^2(t)}{u^2(t)} - \frac{P(t)}{u(t)}\left\{\frac{d}{dt}\left(\dot u(t)P(t)\right) - Q(t)u(t)\right\}$$

$$= w^2(t) - 0$$

Thus we can write,

$$\int_{t_1}^{t_2} \left(P(t)\, \dot{h}^2(t) + Q(t)\, h^2(t) \right) dt$$

$$= \int_{t_1}^{t_2} \Big(P(t)\, \dot{h}^2(t) + 2\, w(t)\, h(t) \dot{h}(t) + \left(Q(t) + \dot{w}(t) \right) h^2(t) \Big) dt$$

$$= \int_{t_1}^{t_2} P(t) \left(\dot{h}(t) + \frac{w(t)\, h(t)}{P(t)} \right)^2 dt$$

$$\geq 0 \qquad \forall\, h(t) \qquad h(t_1) = h(t_2) = 0 .$$

Suppose this expression vanishes for some h.

Then $\dot{h}(t) + \dfrac{w(t)\, h(t)}{P(t)} \equiv 0 \qquad t \in [t_1, t_2]$

Setting $h(t_1) = 0$ we find that $\dot{h}(t_1) = 0$.

By uniqueness of solutions to ode.'s

$h(t) \equiv 0$.

$\Rightarrow$ Positive definiteness. $\blacksquare$