Adjoint equations

For the homogeneous system $\dot{x}(t) = A(t)x(t)$, we have the associated **adjoint system**

$$\dot{p}(t) = -A'(t)p(t).$$

From

$$\begin{aligned}\frac{d}{dt}\left(p'(t)x(t)\right) &= \dot{p}'(t)\,x(t) + p'(t)\,\dot{x}(t) \\ &= \left(-A'(t)p(t)\right)'x(t) + p'(t)\left(A(t)x(t)\right) \\ &\equiv 0\end{aligned}$$

it follows that

$$p'(t)\,x(t) \equiv p'(t_0)\,x(t_0) \qquad \forall t.$$

Moreover, writing $p(t) = \Phi_{-A'}(t,t_0)\,p(t_0)$ we get

$$p'(t_0)\,\Phi'_{-A'}(t,t_0)\,\Phi_A(t,t_0)\,x(t_0) = p'(t_0)\,x(t_0) \qquad \forall t$$

Since this is true for arbitrary $x(t_0)$, $p(t_0)$ it follows that,

$$\Phi'_{-A'}(t,t_0)\,\Phi_A(t,t_0) = \mathbb{1} \qquad \forall t$$

$$\Leftrightarrow \qquad \Phi_{-A'}(t,t_0) = \Phi'_A(t_0,t) \qquad \forall t.$$

From this we also have a corollary;

$$\begin{aligned}\frac{d}{dt}\Phi_A(t_0,t) &= \frac{d}{dt}\left(\Phi'_{-A'}(t,t_0)\right) \\ &= \left(-A'\,\Phi_{-A'}(t,t_0)\right)' \\ &= -\Phi'_{-A'}(t,t_0)\,A(t) \\ &= -\Phi_A(t_0,t)\,A(t)\end{aligned}$$

$\square$

Canonical Equations

Consider the linear time-varying system (canonical equations)

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t)B'(t) \\ -L(t) & -A'(t) \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \qquad (C)$$

evolving on $\mathbb{R}^{2n}$. (Assume $L(t) \equiv L'(t)$). Let $H(t,x,p)$ denote the function

$$H: \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$$
$$(t, x, p) \mapsto H(t,x,p)$$

$$= \frac{1}{2} x' L(t) x + p' A(t) x - \frac{1}{2} p' B(t) B'(t) p$$

Define the gradient of H, $\nabla H = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{pmatrix}$.

$$\nabla H = \begin{pmatrix} L(t) & A'(t) \\ A(t) & -B(t)B'(t) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

We can then rewrite the given system (C) in the form

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathbb{J} \, \nabla H \qquad \text{where } \mathbb{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

is a skew-symmetric invertible matrix. Along trajectories of (C) the derivative of H w.r.t. time can be computed using chain rule:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial p_i} \dot{p}_i$$
$$= \frac{\partial H}{\partial t} + (\nabla H)' \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \frac{\partial H}{\partial t} + (\nabla H)' \mathbb{J} \, \nabla H$$

Since $\mathbb{J}$ is skew, the second term on the right is identically zero. If further, the parameters A, B, L are time-invariant, then $\frac{\partial H}{\partial t} = 0$ and $\frac{dH}{dt} = 0$

$\Rightarrow$ H = constant.

What are canonical equations (C) good for?

For one thing, solving Riccati equations.

<u>Lemma 1</u>

Let $\Phi(t, t_0)$ denote the $2n \times 2n$ transition matrix for (C). Partition into blocks: $\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$ of size $n \times n$. Then:

(a) If $\Pi(t, Q, t_1) = \left(\Phi_{22}(t_1, t) - Q\,\Phi_{12}(t_1, t)\right)^{-1} \left(Q\,\Phi_{11}(t_1, t) - \Phi_{12}(t_1, t)\right)$

or equivalently, if

(b) $\Pi(t, Q, t_1) = \left(\Phi_{21}(t, t_1) + \Phi_{22}(t, t_1)Q\right)\left(\Phi_{11}(t, t_1) + \Phi_{12}(t, t_1)Q\right)^{-1}$

then $\Pi(t_1, Q, t_1) = Q$ <u>and</u>, Π satisfies the Riccati equation,

$$\dot{\Pi} = -A'\Pi - \Pi A + \Pi B B' \Pi - L$$

assuming that the indicated inverses exist.

<u>Proof</u>: Let, $[X(t), -P(t)] \triangleq [Q, -\mathbb{1}] \begin{bmatrix} \Phi_{11}(t_1, t) & \Phi_{12}(t_1, t) \\ \Phi_{21}(t_1, t) & \Phi_{22}(t_1, t) \end{bmatrix}$

Clearly, in part (a) above, the r.h.s $= P^{-1}(t)\, X(t)$.

To verify that $\Pi(t, Q, t_1) = P^{-1}(t)\,X(t)$ satisfies the Riccati Equation, differentiate:

$$\frac{d}{dt}(P^{-1}X) = -P^{-1}\dot{P}P^{-1}X + P^{-1}\dot{X}$$

From the definition of $[X, P]$ we see

$$[\dot{X}, -\dot{P}] = [Q, -\mathbb{1}]\frac{d}{dt}\Phi(t_1, t)$$

$$= [Q, -\mathbb{1}]\frac{d}{dt}\Phi^{-1}(t, t_1)$$

$$= [Q, -\mathbb{1}]\left(-\Phi^{-1}(t, t_1)\right)\frac{d\Phi}{dt}(t, t_1)\,\Phi^{-1}(t, t_1)$$

$$= -[Q, -\mathbb{1}]\,\Phi(t_1, t)\begin{pmatrix} A(t) & -B(t)B'(t) \\ -L(t) & -A'(t) \end{pmatrix}$$

$$= -[X, -P]\begin{pmatrix} A & -BB' \\ -L & -A' \end{pmatrix}$$

$$\Rightarrow \quad \dot{X} = -XA - PL$$

$$\dot{P} = -XBB' + PA'$$

$$\Rightarrow \quad \frac{d}{dt}(P^{-1}X) = -P^{-1}(-XBB' + PA')P^{-1}X + P^{-1}(-XA - PL)$$

$$= P^{-1}XBB'P^{-1}X - A'P^{-1}X - P^{-1}XA - L$$

which is what we set out to prove.

Also $P^{-1}(t)X(t)\Big|_{t=t_1} = Q$ since $\Phi(t_1, t_1) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$.

Proof of part (b)

Define $\begin{pmatrix} \tilde{X}(t) \\ \tilde{P}(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t, t_1) & \Phi_{12}(t, t_1) \\ \Phi_{21}(t, t_1) & \Phi_{22}(t, t_1) \end{pmatrix}\begin{pmatrix} \mathbb{1} \\ Q \end{pmatrix}$.

Clearly, in part (b) above r.h.s $= \tilde{P}(t)\,\tilde{X}(t)^{-1}$. Rest of the steps similar to the steps in part (a) proof. ▨

Remark Matrices of the form

$$P = \begin{pmatrix} A & Q \\ R & -A' \end{pmatrix}$$

where each of the blocks is $n \times n$ and $Q = Q'$, $R = R'$ are called infinitesimally symplectic or hamiltonian matrices. They satisfy the identity.

$$P'\mathbb{J} + \mathbb{J}P = 0$$

Necessary Conditions for Optimality

Theorem For the system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $x(t_0) = x_0$, let $u(t)$ be one of the following controls

(a) $u_0(t) = -B'(t)\Phi_A'(t_0,t)\xi$ where ξ satisfies

$$W(t_0,t_1)\xi = x_0 - \Phi_A(t_0,t_1)x_1$$

(b) $u_1(t) = -B'(t)\Pi(t,Q,t_1)x(t)$

$$\dot{\Pi}(t) = -A'(t)\Pi(t,Q,t_1) - \Pi(t,Q,t_1)A(t) - L(t) + \Pi(t,Q,t_1)B(t)B'(t)\Pi(t,Q,t_1)$$

$\Pi(t_1,Q,t_1) = Q$ and $L(t) = L'(t)$ $\forall t$

(c) $u_2(t) = -B'(t)\Pi(t,K_1,t_1)x(t) + v(t)$

$$\dot{\Pi} = -A'\Pi - \Pi A + \Pi BB'\Pi$$

$\Pi(t_1,K_1,t_1) = K_1$ and v such that it minimizes $\int_{t_0}^{t_1} v'(\sigma)v(\sigma)\,d\sigma$ for $\dot{x}(t) = (A(t) - B(t)B'(t)\Pi(t,K_1,t_1))x(t)$, $x(t_0) = x_0$; $x(t_1) = x_1$

Then, there exists a vector function $p(t)$ (the co-state) such that

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -L & -A' \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} . \quad x(t_0) = x_0$$

and $u(t) = -B'(t)\, p(t)$.

~~Proof~~ (a) Let $p(t) = \Phi_A'(t_0, t)\, \xi$

Then $\dot{p} = -A'(t)\, p$

with $p_0 = \xi$ (from page 1 of this lecture).

Substituting u_0 in the state equation, we get

$$\dot{x} = Ax - BB'p$$

Picking $L \equiv 0$ ensures that
$$\dot{p} = -A'p = -Lx - A'p$$

This completes the proof of part (a).

(b) Let $p(t) = \Pi(t, Q, t_1)\, x(t)$

Then substituting u_1 in the state equation we get,

$$\dot{x} = Ax - BB'p \quad \checkmark$$

We need to show

$$\dot{p} = -Lx - A'p$$

Differentiate $\Pi(t, Q, t_1)\, x(t)$ to get,

$$\dot{p} = \dot{\Pi} x + \Pi \dot{x} = (-A'\Pi - \Pi A - L)x + \Pi BB'\Pi + \Pi(Ax - BB'\Pi x)$$

$$= -A'p - Lx$$

The boundary condition on π turns into

$$p(t_1) = Qx(t_1).$$

(c) Left as an exercise ▨

<u>Remark</u> We postpone discussion of the infinite horizon optimal control problem and associated algebraic Riccati equations.

<u>Using the Canonical Equations</u>

From proof of part (a) of the Theorem, it is clear that solving (C) for (x_0, p_0), initial conditions, sweeps out a 'bundle' of state/costate trajectories as p_0 is varied. Only p_0 s.t. $W(t_0, t_1)p_0 = x_0 - \underset{A}{\Phi}(t_0, t_1)x_1$ will produce trajectory/trajectories satisfying end-point conditions. End-point error associated to a given p_0 can be used to correct p_0. Similar remarks apply to cases (b) & (c).

<u>Analogues of (C)</u> play a central role in general optimal control problems. (not necessarily linear systems with quadratic cost functionals). We will encounter these later.