This is a lecture on calculus on linear spaces — not necessarily finite dimensional.

We denote vector spaces as X, Y etc. Linear operators/maps from X to Y themselves form a vector space denoted as $L(X,Y)$. A norm on X is a non-negative function $\|\cdot\| : X \to \mathbb{R}_+$ satisfying axioms

(a) $\|\alpha x\| = |\alpha| \, \|x\| \qquad \alpha \in \mathbb{R}, \; x \in X$

(b) $\|x\| = 0 \;\Rightarrow\; x = 0$

(c) $\|x+y\| \le \|x\| + \|y\|$ (triangle inequality)

A linear map $A : X \to Y$ between normed linear spaces is said to be bounded if there exists a constant $c \geq 0$ such that $\|Ax\| \le c\|x\| \quad \forall\, x \in X$.

If such a 'c' exists then '$c+b$' would do as well for any $b \geq 0$.

We define norm of A, denoted $\|A\|$ to be

$$\inf \{ c : \|Ax\| \le c\|x\| \quad \forall\, x \in X \}$$

The property of boundedness is preserved under addition of linear maps:

A_1 such that $\|A_1 x\| \le c_1 \|x\| \quad \forall\, x \in X$

A_2 such that $\|A_2 x\| \le c_2 \|x\| \quad \forall\, x \in X$

$$\begin{aligned}
\|(A_1 + A_2)x\| &= \|A_1 x + A_2 x\| \\
&\le \|A_1 x\| + \|A_2 x\| && \text{(triangle ineq.)} \\
&\le c_1\|x\| + c_2\|x\| && \text{(by hypothesis)} \\
&\le (c_1 + c_2)\|x\|
\end{aligned}$$

One also has the equivalent definitions

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\|$$

<u>Proposition 1</u> A linear operator $A : X \to Y$ is bounded <u>iff</u> it is continuous.

<u>Proof</u> ($\Rightarrow$) Let A be bounded. Then $\|Ax - Ay\| = \|A(x-y)\|$

$$\leq \|A\| \cdot \|x-y\|$$

$\|A\| \neq 0$ (assumption)

Thus given $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{2\|A\|}$. Then,

$$\begin{aligned} \|x-y\| < \delta \Rightarrow \quad \|Ax - Ay\| &\leq \|A\| \cdot \|x-y\| \\ &< \|A\| \frac{\varepsilon}{2\|A\|} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Hence A is continuous.

($\Leftarrow$) Assume A is continuous and linear. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow \|Ax\| < \varepsilon. \qquad \left(\begin{array}{c}\text{by continuity} \\ \text{at } x=0\end{array}\right)$$

For any $x \neq 0$

$$\begin{aligned} \left\| \frac{\delta/2}{\|x\|} \cdot x \right\| &= \frac{|\delta/2|}{\|x\|} \cdot \|x\| \\ &= \frac{\delta}{2} < \delta \end{aligned}$$

and ~~$\|Ax\| = \|A$~~

$$\begin{aligned} Ax &= A\left(\left(\frac{\delta/2}{\|x\|}\right)^{-1} \frac{\delta/2}{\|x\|} \cdot x \right) \\ &= \frac{\|x\|}{\delta/2} A\left(\frac{\delta/2}{\|x\|} \cdot x \right) \qquad (\text{linearity of } A) \end{aligned}$$

Thus $\|Ax\| = \dfrac{\|x\|}{\delta/2} \cdot \left\| A\left(\dfrac{\delta}{2} / \|x\| \cdot x\right) \right\| < \dfrac{\|x\|}{\delta/2} \varepsilon$ (continuity of A)

$$= \frac{2\varepsilon}{\delta} \|x\|$$

Thus $\|Ax\| < \frac{2\varepsilon}{\delta} \|x\| \quad \forall\, x \in X$

$\Rightarrow$ A is bounded. $\blacksquare$

The collection of all bounded linear operators $A: X \to Y$ is a vector space (as shown at the bottom of page 1). We denote this as $\mathcal{B}(X, Y)$. Show that the operator norm $\|A\|$ defined ~~by~~ above, is ~~a~~ norm on the space $\mathcal{B}(X,Y)$ (satisfying the axioms (a) (b) (c) on page 2) — you need to verify the triangle inequality.

Also show that $\|AB\| \leq \|A\| \cdot \|B\|$ for $A, B \in \mathcal{B}(X, X)$

Examples of normed linear spaces and associated operator norms.

(i) $X = \mathbb{R}^n$. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$; $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ are different norms on same space

(ii) $Y = \mathbb{R}^m$, $X = \mathbb{R}^n$ $\|\cdot\|$ on X and Y are $\|\cdot\|_1$

$A: X \to Y$ (matrix multiplication operator)

$$\|Ax\| = \sum_{i=1}^{m} \left|\sum_{j=1}^{n} a_{ij} x_j\right|$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} x_j| \qquad \text{(triangle inequality)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|\, |x_j|$$

$$\leq \sum_{i=1}^{m} \max_{1 \leq j \leq n} |a_{ij}| \sum_{j=1}^{n} |x_j|$$

$$= \left(\sum_{i=1}^{m} \max_{1 \le l \le k} |a_{il}|\right) \|x\|_1$$

So A is bounded. What is $\|A\|$?

(iii) $X = C[0,1]$ = space of continuous real valued functions on the interval $[0,1]$.

Let the norm on X be

$$\|x\|_1 \triangleq \int_0^1 |x(t)| \cdot dt$$

$A : X \to \mathbb{R}$

$x \mapsto x(\tfrac{1}{2})$ = value of the function x at $t = \tfrac{1}{2}$.

A is a linear map. Verify.

<u>Question</u>: Is A a bounded linear operator?

(iv) $X = C[0,1]$ = space as in (iii). Let norm on X be $\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|$

A same as in (iii). It is bounded since,

$$\|Ax\| = |x(\tfrac{1}{2})| \le \max_{0 \le t \le 1} |x(t)| = \|x\|_\infty$$

$\Rightarrow \|A\| = 1$.

We denote a sequence $x_1, x_2, x_3, \dots, x_n, \dots$ in a normed linear space as $\{x_n\} \subset X$.

<u>Definition</u> (Convergence) A sequence $\{x_n\} \subset X$ is said to converge to $x^* \in X$ if $\lim_{n \to \infty} \|x_n - x^*\| = 0$.

(i.e. given $\varepsilon > 0$ there exists a positive integer $N > 0$ such that $\| x_n - x^* \| < \varepsilon \quad \forall \; n > N$).

<u>Definition</u> (Cauchy sequence)

A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $\lim\limits_{\substack{n \to \infty \\ m \to \infty}} \| x_n - x_m \| = 0$.

<u>Proposition 2</u> Convergent $\Rightarrow$ Cauchy.

<u>Proof</u>: Given $\varepsilon > 0$ $\exists$ N integer > 0 s.t.

$$\| x_n - x^* \| < \varepsilon/2 \quad \forall \; n > N$$

But
$$\begin{aligned} \| x_n - x_m \| &= \| (x_n - x^*) - (x_m - x^*) \| \\ &\le \| x_n - x^* \| + \| x_m - x^* \| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall \; n, m > N \\ &= \varepsilon \end{aligned}$$

$\therefore \Rightarrow \{x_n\}$ is Cauchy. ▨

Converse of the above proposition is <u>not true</u> <u>in general</u>. However, if X is a normed linear space such that, <u>every</u> Cauchy sequence in X is also a convergent sequence in X, we say that X is a <u>complete</u> normed linear space — a <u>Banach</u> space

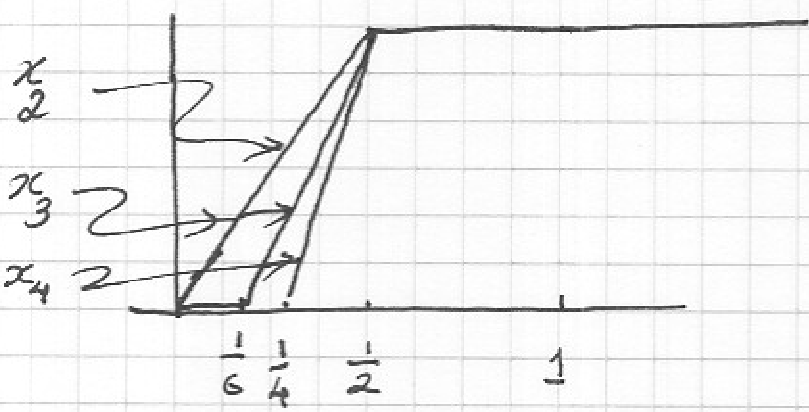
A fundamental property of $X = \mathbb{R}$ with $\|x\| = |x|$ is that it is a complete normed linear space. But,

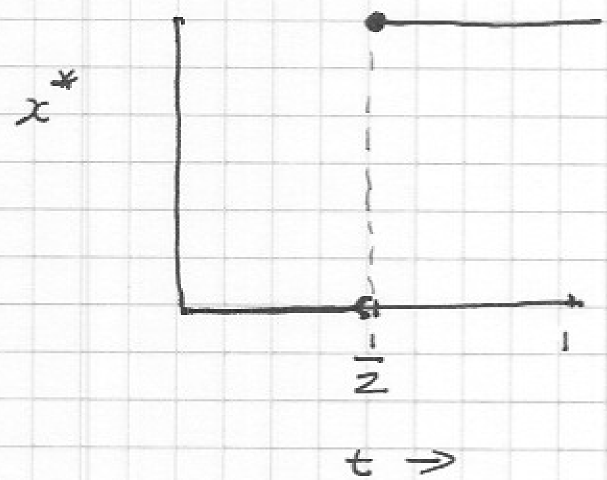
$X = C[0,1]$ with $\|x\|_1 = \int_0^1 |x(t)|\,dt$ is not

a single Cauchy sequence that does not converge in X.

Consider $n = 2, 3, 4, \ldots$

$$x_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \frac{1}{2} - \frac{1}{n} \le t < \frac{1}{2} \\ 1 & t \ge \frac{1}{2} \end{cases}$$





$$x^*(t) \; \cancel{x_\infty(t)} = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1 \end{cases}$$

We claim $x_n \to x^*$ in $\|\cdot\|_1$.

Proof: $\|x_n - x^*\|_1 = \int_0^1 |x_n(t) - x^*(t)|\,dt$

$$= \int_0^{\frac{1}{2}-\frac{1}{n}} + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1}$$

$$= 0 + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \left(nt - \frac{n}{2} + 1\right) dt + \int_{\frac{1}{2}}^{1} (1-1)\, dt$$

$$= \left(\frac{nt^2}{2} - \frac{nt}{2} + t\right)\Bigg|_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}}$$

$$= \frac{1}{2n} \to 0 \quad \text{as } n \to \infty \qquad \blacksquare$$

But $x^* \notin X$ since x^* is discontinuous at $\frac{1}{2}$. Thus the sequence $\{x_n\}$ does not converge to anything in X.

Separately, verify that $\{x_n\}$ is Cauchy. Conclude that X is not complete under the norm $\|\cdot\|_1$. But one can show that $(X, \|\cdot\|_\infty)$ where $\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|$ is a complete normed linear space. Completeness depends on the chosen norm.

We now proceed to the study of derivatives of maps between general linear spaces.

<u>Derivatives</u> Let X be a vector space and let Y be a normed linear space with norm $\| \ \|$. Let $T : X \to Y$ be an operator / map — not necessarily linear.

If the limit

$$\delta T(x;h) = \lim_{\alpha \to 0} \frac{1}{\alpha}\left(T(x+\alpha h) - T(x)\right)$$

exists, then $\delta T(x;h)$ is called the <u>Gateaux differential</u> of T at x with increment h. If the limit exists for every $h \in X$, the map T is said to be <u>Gateaux differentiable</u> at x. In that case the operator

$$h \mapsto \delta T(x;h)$$

is the <u>Gateaux derivative</u> operator at $x \in X$.

<u>Q</u>: Does it have to be a linear operator?

We also refer to $\delta T(x;h)$ as the <u>first variation</u> of T at x with increment h.

<u>Example</u> $X = \mathbb{R}^n$, $Y = \mathbb{R}$ $\quad f : \mathbb{R}^n \to \mathbb{R}$

$$\delta f(x;h) = \lim_{\alpha \to 0} \frac{f(x+\alpha h) - f(x)}{\alpha}$$

$$= \left.\frac{d}{d\alpha} f(x+\alpha h)\right|_{\alpha=0}$$

$=$ directional derivative of f along h at x.

If f has continuous first partials w.r.t. each variable x_i, then the chain rule applies and

$$\begin{aligned} \delta f(x;h) &= \frac{d}{d\alpha} f(x+\alpha h) \Big|_{\alpha=0} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+\alpha h)\, h_i \Big|_{\alpha=0} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot h_i \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \end{aligned}$$

Thus we think of the Gateaux derivative as the operator of multiplication by the row vector $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$, clearly a linear operator. $\boxtimes$

<u>Example</u> $\quad X = C[0,1]$

$$f : X \to \mathbb{R} \qquad f(x) = \int_0^1 g(x(t), t)\, dt$$

for a specified function of two variables g.

Assume that $g_x = \frac{\partial g}{\partial x}$ = partial of g w.r.t. the first argument, exists and is continuous w.r.t x and t. Then,

$$\delta f(x;h) = \frac{d}{d\alpha}\int_0^1 g(x(t)+\alpha h(t), t)\,dt \Big|_{\alpha=0}$$

$$= \int_0^1 \frac{d}{d\alpha} g(x(t)+\alpha h(t), t)\,dt \Big|_{\alpha=0}$$

$$= \int_0^1 g_x(x(t)+\alpha h(t), t)\,h(t)\,dt \Big|_{\alpha=0}$$

$$= \int_0^1 g_x(x(t), t)\,h(t)\,dt$$

again linear in h ▨

Example

$T: \mathbb{R}^n \to \mathbb{R}^m$ with continuous first partials

Then $\delta T(x;h) = \left(\frac{\partial T}{\partial x}\right) h$

matrix of first partials ▨

The concept of Gateaux derivative is a weak concept. It does not require X to have a norm. Suppose X does carry a norm. Then we have a stronger notion

Fréchet differential.. → definition?

Let $U \subset X$ be an open subset of X containing the point x_0. Suppose there exists a linear bounded map $DT(x;\cdot) : X \to Y$

$$h \mapsto DT(x;h)$$

such that

$$\lim_{\|h\| \to 0} \frac{\|T(x+h) - T(x) - DT(x;h)\|_Y}{\|h\|_X} = 0.$$

Then T is said to be Fréchet differentiable at $x \in U \subset X$ and $DT(x;h)$ is called the Fréchet differential of T at x with increment h. The operator $DT(x;\cdot)$ is the Fréchet derivative.

Proposition 3 (uniqueness)

If $T : X \to Y$ has a Fréchet differential at $x \in X$ then it is unique.

Proof : Exercise $\blacksquare$

Proposition 4 ($F = G$)

If the Fréchet differential $DT(x;h)$ exists at x, with increment h, then so does the Gateaux differential $\delta T(x;h)$ and they must be equal.

Proof By the definition of Fréchet differential, for fixed h

$$\lim_{\alpha \to 0} \frac{\| T(x+\alpha h) - T(x) - DT(x;\alpha h) \|}{\alpha} = 0$$

By linearity of $DT(x;\alpha h)$ w.r.t αh it follows that $$\lim_{\alpha \to 0} \frac{T(x+\alpha h) - T(x)}{\alpha} = DT(x; h)$$

i.e. $$\delta T(x; h) = DT(x; h)$$ $\blacksquare$

<u>Proposition 5</u> (continuity from differentiability)

If $T : U \subset X \to Y$ is Fréchet differentiable at x then T is continuous at x, (Here $x \in U$).

<u>Proof</u> Given $\epsilon > 0$, there is a ball centered at x of radius [illegible]

provided ε is sufficiently small

For $x + h \in B_{\varepsilon}(x)$,

$$\| T(x+h) - T(x) - DT(x; h) \| \le \epsilon \| h \|.$$

Thus
$$\begin{aligned}
\| T(x+h) - T(x) \| &\le \varepsilon \|h\| + \| DT(x; h) \| \\
&\le \varepsilon \|h\| + \| DT(x; \cdot) \| \, \|h\| \\
&\le \left(\varepsilon + \| DT(x; \cdot) \| \right) \|h\| \\
&= M \|h\|
\end{aligned}$$ $\blacksquare$

<u>Example</u> Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous first partial derivatives at $x_0 \in \mathbb{R}^n$. Then the differential

$$\delta f(x_0; h) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)_{|x = x_0} \cdot h_i$$ is <u>the</u> Fréchet differential.

Some notation

Since $DT(x;h)$ is linear in h, it is customary to write

$$DT(x;h) = DT(x)\,h$$

We call $DT(x)$ the Fréchet derivative operator at x. It is a <u>bounded</u> linear operator by definition.

Another Example

$f: \mathbb{R}^n \to \mathbb{R}$ $\qquad f(x) = x'Qx$

where $Q = Q'$. Then f is Gateaux and Fréchet differentiable at any $x \in \mathbb{R}^n$ and

$$Df(x) = 2x'Q$$

To see this –

$$\frac{d}{d\alpha} f(x+\alpha h) = \frac{d}{d\alpha}(x+\alpha h)'Q(x+\alpha h)$$

$$= \frac{d}{d\alpha}\{x'Qx + \alpha h'Qx + \alpha x'Qh + \alpha^2 h'Qh\}$$

$$= 2x'Qh + 2\alpha h'Qh$$

$$\delta f(x;h) = \frac{d}{d\alpha} f(x+\alpha h)\Big|_{\alpha=0} = 2x'Qh = Df(x)\,h$$

Hence $Df(x) = 2x'Q$