Let X, Y, Z be normed spaces and let

$$T : X \times Y \longrightarrow Z$$
$$(x, y) \longmapsto z = T(x, y)$$

be a nonlinear map.

$D_1 : X \to Z$ a bounded linear map is the partial Fréchet derivative at (x,y) with respect to the first factor if

$$\lim_{h \to 0} \frac{\| T(x+h, y) - T(x,y) - D_1 \cdot h \|_Z}{\|h\|_X} = 0$$

Similarly $D_2 : Y \to Z$ is the partial Fréchet derivative at (x,y) with respect to the second factor if

$$\lim_{w \to 0} \frac{\| T(x, y+w) - T(x,y) - D_2 \cdot w \|_Z}{\|w\|_Y} = 0$$

When $X = \mathbb{R}^n$ $Y = \mathbb{R}^m$ $Z = \mathbb{R}^p$

We can write

$$D_1 = D_1(x,y) = \left[\frac{\partial T_i}{\partial x_j} \right]_{p \times n}$$

$$D_2 = D_2(x,y) = \left[\frac{\partial T_i}{\partial y_j} \right]_{p \times m}$$

IMPLICIT FUNCTION THEOREM ~~(IFT)~~ (IFT)

Let $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ have a Fréchet derivative $Df(\theta, x)$ for each $(\theta, x) \in S$ an open subset of $\mathbb{R}^m \times \mathbb{R}^n$.

Assume that $Df(\theta, x)$ is continuous on U.
Let (θ_0, x_0) be such that

$$f(\theta_0, x_0) = 0 \qquad \underline{\text{and}}$$

the partial Fréchet derivative

$$\underset{n\times n}{D_2 f}(\theta_0, x_0) \quad \text{is nonsingular.}$$

Then there exist <u>open neighborhoods</u> U of θ_0 in $\mathbb{R}^m$ and V of x_0 in $\mathbb{R}^n$ such that, for each $\theta \in U$, the equation

$$f(\theta, x) = 0$$

has a unique solution $x \in U$. Moreover this solution can be given as

$$x = g(\theta)$$

where g is Fréchet differentiable near θ_0 and $(Dg)(\theta)$ is continuous in θ at $\theta = \theta_0$.

<u>Remarks</u>: · This theorem extends to Banach spaces.

Theorem 1 : Let X be a vector space. Let $f_i : X \to \mathbb{R}$, $i = 1, 2, 3, \cdots, n$ be n linearly independent linear functionals. Then there exist vectors $x_1, x_2, \cdots, x_n \in X$ such that the matrix

$$\left[f_i(x_j) \right]$$

is nonsingular.

Proof by induction

For $n=1$ the result is true since given a nontrivial functional $f : X \to \mathbb{R}$ there exists $x \in X$ such that $f(x) \neq 0$ ← typo

induction hypothesis : suppose the theorem holds for n, we aim to prove it is true for $n+1$

By hypothesis $\left[f_i(x_j) \right]_{n \times n}$ is nonsingular

Let $f_{n+1} : X \to \mathbb{R}$ be a linear functional linearly independent of $f_1, f_2, \cdots, f_n$. For any $x_{n+1} \in X$

$$A_{n+1} = \left[f_i(x_j) \right]_{(n+1) \times (n+1)}$$

$$= \left[\begin{array}{c|c} A_n & \begin{matrix} f_1(x_{n+1}) \\ \vdots \\ f_n(x_{n+1}) \end{matrix} \\ \hline f_{n+1}(x_1) \cdots f_{n+1}(x_n) & f_{n+1}(x_{n+1}) \end{array} \right] = \left[\begin{array}{c|c} A_n & b \\ \hline c & d \end{array} \right]$$

By the Schur formula

$$\det(A_{n+1}) = \det(A_n)\det(d - c A_n^{-1} b)$$

(recall A_n is invertible by hypothesis)

But $\det(d - c A_n^{-1} b)$

$$= d - c A_n^{-1} b$$

$$= f_{n+1}(x_{n+1}) - (f_{n+1}(x_1), \dots, f_{n+1}(x_n)) A_n^{-1} \begin{pmatrix} f_1(x_{n+1}) \\ \vdots \\ f_n(x_{n+1}) \end{pmatrix}$$

$$= \left(f_{n+1} + \sum_{i=1}^{n} \alpha_i f_i\right)(x_{n+1})$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$

$$= -(f_{n+1}(x_1), \dots, f_{n+1}(x_n))\, A_n^{-1}.$$

Since $f_1, f_2, \dots, f_{n+1}$ are linearly independent the linear combination,

$$f_{n+1} + \sum_{i=1}^{n} \alpha_i f_i$$

is a <u>nontrivial</u> functional. Hence there exits x_{n+1} such that $\left(f_{n+1} + \sum_{i=1}^{n} \alpha_i f_i\right)(x_{n+1}) \neq 0$.

$\Rightarrow \det(A_{n+1}) \neq 0$ ▨

<u>Remark</u> It can be shown that the vectors $x_1, x_2, \ldots, x_n$ in the theorem are <u>necessarily</u> linearly independent. To see this,

Let $A_n = (f_i(x_j))_{n \times n}$ matrix with ijth element $= f_i(x_j)$

Let $B_n = A_n^{-1} = (b_{ij})$

Hence $A_n B_n = (f_i(x_j))(b_{ij}) = \mathbb{1}$ the identity matrix.

$$\cancel{\Rightarrow \sum_{i=1}^{n} b_{ij} f_j(x_i) = 0 \quad j=1,2,\ldots,n}$$

$$\Rightarrow \sum_{k=1}^{n} f_i(x_k)\, b_{kj} = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\Leftrightarrow f_i\left(\sum_{k=1}^{n} b_{kj}\, x_k\right) = \delta^i_j$$

$$\leftrightarrow f_i(\tilde{x}_j) = \delta^i_j$$

where $\tilde{x}_j = \sum_{k=1}^{n} b_{kj}\, x_k \qquad j = 1, 2, \ldots, n$

Suppose $\exists\ \beta_i$ s.t. $\sum_{j=1}^{n} \beta_j \tilde{x}_j = 0$.

$$\text{Then } 0 = f_i\left(\sum_{j=1}^{n} \beta_j \tilde{x}_j\right) = \sum_{j=1}^{n} \beta_j f_i(\tilde{x}_j) = \sum_{j=1}^{n} \beta_j \delta^i_j = \beta_i$$

$i = 1, 2, \ldots, n.$

$\Rightarrow \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ are linearly independent

Since $\tilde{x}_j$ are related to x_k etc by a nonsingular matrix B, it follows that $x_1, x_2, \ldots, x_n$ are linearly independent. ▨

When $\dim(X) = n$, then $\tilde{x}_1, \ldots, \tilde{x}_n$ constitute a basis for X and $f_1, f_2, \ldots, f_n$ constitute the <u>dual</u> basis for the space X^* of linear functionals on X.

<u>Theorem 2</u>: Let $f, f_1, f_2, \ldots, f_n$ be linear functionals on X. Suppose, for each x such that $f_i(x) = 0$ $i = 1, 2, \ldots, n$, we have $f(x) = 0$. Then $f = \sum_{i=1}^{n} \alpha_i f_i$ for some α_i

Proof There is <u>no loss of generality in assuming</u> that the functionals $f_1, f_2, \ldots, f_n$ are linearly independent. → simplified version

$\boxed{\text{Suppose } \dim X = m < \infty.}$ Then $m \geq n$

Then $\dim(X^*) = m$.

Complete the set $\{f_1 f_2, \ldots, f_m\}$ with functionals $\{f_{m+1}, f_{m+2}, \ldots, f_n\}$ so as to get a basis for X^*. Let $\{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m\}$ be the dual basis for X. Thus

$$f_i(x_j) = \delta_j^i \qquad i \neq j = 1, 2, \ldots, m.$$

Then $f = \sum_{i=1}^{m} \alpha_i \cdot f_i$

$\operatorname{Ker}(f_i) = \operatorname{span}\{x_j : j=1,2,\dots,m.\ j \neq i\}$

$i=1,2,\dots,~~m~~\ n$

Then $\bigcap_{i=1}^{n} \operatorname{Ker}(f_i) = \operatorname{span}\{x_j : j=n+1, n+2, \dots, m\}$

By hypothesis $\bigcap_{i=1}^{n} \operatorname{Ker}(f_i) \subset \operatorname{Ker}(f)$

$\Rightarrow \quad f(x_j) = 0 \qquad j = n+1, n+2, \dots, m$

$$\Rightarrow \left(\sum_{i=1}^{m} \alpha_i f_i\right)(x_j)$$

$$= \sum_{i=1}^{m} \alpha_i f_i(x_j) = 0, \quad j = n+1, n+2, \dots, m$$

By dual basis property

$$\sum_{i=1}^{m} \alpha_i f_i(x_j) = \alpha_j$$

$\Rightarrow \alpha_j = 0 \qquad j = n+1, n+2, \dots, m.$

$$\Rightarrow \quad f = \sum_{i=1}^{n} \alpha_i f_i \qquad \blacksquare$$

<u>EXERCISE</u> develop a proof without the assumption of finite dimensionality of X

<u>Hint</u>: Consider $\tilde{f} = \sum_{i=1}^{n} f(x_i) f_i$

show that $\tilde{f} = f$ on

$V = \operatorname{span}\{x_1, x_2, \dots, x_n\}$ where x_i $i=1,2,\dots,n$ is the dual basis of f_j $j=1,2,\dots,n$. Extend this to $\tilde{f} = f$ on all X