Polynomial Methods

Division of polynomial by polynomial

Let $a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$ and $b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m$ be two scalar polynomials with $b_0 \neq 0$, $n \geq m$.

There exist unique polynomials $q(s)$ (the QUOTIENT) and $r(s)$ (the REMAINDER) such that

$$a(s) = q(s)\, b(s) + r(s)$$

<u>and</u> $\deg r(s) < \deg b(s)$

The algorithm which accomplishes division is the EUCLIDEAN algorithm.

$$\begin{aligned}
a(s) &= a_0 s^n + a_1 s^{n-1} + \cdots + a_n \\
&= a_0 b_0^{-1} s^{n-m} (b_0 s^m + \cdots + b_m) \\
&\qquad + (a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n) \\
&\qquad - (b_1 s^{m-1} + b_2 s^{m-2} \cdots + b_m) a_0 b_0^{-1} s^{n-m} \\
&= a_0 b_0^{-1} s^{n-m}\, b(s) + r^{(1)}(s)
\end{aligned}$$

where

$$\begin{aligned}
r^{(1)}(s) = {} & (a_1 - b_1 a_0 b_0^{-1}) s^{n-1} \\
& + (a_2 - b_2 a_0 b_0^{-1}) s^{n-2} \\
& + \cdots + \cancel{a_{n-m} - b_m a_0 b_0} \\
& (a_m - b_m a_0 b_0^{-1}) s^{n-m} \\
& + a_{m+1} s^{n-m-1} + \cdots + a_n
\end{aligned}$$

and $\deg(r^{(1)}(s)) \leq n-1$.

Thus the above substitution lowers the degree of $r^{(1)}(s)$ by 1. Repeat by dividing $r^{(1)}(s)$ by $b(s)$ to obtain $r^{(2)}(s)$ of degree $\leq (n-2)$, (and $r^{(2)}(s)$ by $b(s)$, etc.) until we end up with ~~the~~ $r(s)$ with degree $r(s) < m$. This is when the algorithm terminates.

Uniqueness follows from observation that if

$$a(s) = q(s)\, b(s) + r(s)$$
$$= \tilde{q}(s)\, b(s) + \tilde{r}(s)$$

where $\deg(r(s)) < \deg(b(s))$ and $\deg(\tilde{r}(s)) < \deg(b(s))$,

then,

$$(q(s) - \tilde{q}(s))\, b(s) = (\tilde{r}(s) - r(s))$$

But r.h.s has degree $< \deg(b(s))$ while l.h.s has degree $\geq \deg(b(s))$, if $q \neq \tilde{q}$ and $r \neq \tilde{r}$. Hence at least $q = \tilde{q}$ or $r = \tilde{r}$.
By the same degree consideration, it must mean $q = \tilde{q}$ and $r = \tilde{r}$ □

Finding g.c.d. of $a(s)$ and $b(s)$.

Without loss of generality, assume $\deg(a(s)) = n \geq \deg(b(s)) = m$

Apply Euclidean division repeatedly as follows:

$$a(s) = q_1(s)\, b(s) + r_1(s) \qquad \deg(r_1(s)) < \deg(b(s))$$

$$b(s) = q_2(s)\, r_1(s) + r_2(s) \qquad \deg(r_2(s)) < \deg(r_1(s))$$

$$r_1(s) = q_3(s)\, r_2(s) + r_3(s) \qquad \deg(r_3(s)) < \deg(r_2(s))$$

$$\vdots$$

$$r_{p-3}(s) = q_{p-1}(s)\, r_{p-2}(s) + r_{p-1}(s)$$

$$r_{p-2}(s) = q_p(s)\, r_{p-1}(s) + 0$$

where at the p^{th} stage remainder $= 0$ since remainder is always lower in degree than the divisor. We say r_{p-1} divides r_{p-2} exactly (denoted by $r_{p-1} \mid r_{p-2}$). By substitution in the previous step, it follows that

$r_{p-1} \mid r_{p-3}, \quad r_{p-1} \mid r_{p-4}, \cdots, r_{p-1} \mid r_1$

$r_{p-1} \mid b$ and hence $r_{p-1} \mid a$.

Thus r_{p-1} is a <u>common factor</u> of $a(s)$ and $b(s)$. We need to <u>prove</u> that r_{p-1} is the g.c.d.

<u>Observe</u>

$$r_1 = 1 \cdot a + (-q_1)\, b$$
$$r_2 = 1 \cdot b + (-q_2)\, r_1$$
$$= 1 \cdot b + (-q_2)\,(1 \cdot a + (-q_1) \cdot b)$$
$$\cancel{= (1 + q_1 q_2)\, b}$$
$$= (-q_2)\, a + (1 + q_1 q_2)\, b$$

$$r_3 = 1 \cdot r_1 - q_3 \cdot r_2$$
$$= 1 \cdot (1 \cdot a + (-q_1) b) + (-q_3)\,((-q_2) a + (1 + q_1 q_2) b)$$
$$= (1 - q_2 q_3) \cdot a + (-q_1 - q_3 - q_1 q_2 q_3)\, b$$

$\vdots$

$$r_{p-1} = x \cdot a + y \cdot b$$

Hence *any* exact divisor of $a(s)$ and $b(s)$ also divides $r_{p-1}(s)$ exactly. But r_{p-1} divides a and b exactly.

Thus $r_{p-1} = g.c.d\,(a, b)$ ▨

We say $a(s)$ and $b(s)$ are *coprime* (or *relatively prime*) if

$g.c.d\,(a, b)$ is a constant, (taken to be $=1$) . Then denote

$$(a, b) \equiv 1$$

Theorem [BEZOUT]

Let $a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$ and $b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m$, $b_0 \neq 0$ and $a_0 \neq 0$. Then

$$(a(s), b(s)) \equiv 1 \qquad \text{(coprimeness)}$$

iff there exist (necessarily unique) polynomials $x(s)$ and $y(s)$ such that

$$x(s)\, a(s) + y(s)\, b(s) \equiv 1$$

and $\deg(x(s)) < m$, $\deg(y(s)) < n$. !

Proof of Bezout's Theorem

($\Rightarrow$) We showed that

$$\text{g.c.d.}\ (a(s), b(s))$$

$$= r_{p-1}(s)$$

$$= x(s)\ a(s) + y(s) b(s)$$

Thus if g.c.d $\equiv 1$ then

$$x(s)\ a(s) + y(s)\ b(s) \equiv 1$$

($\Leftarrow$) Suppose $\exists$ solution to

$$x\ a + y\ b \equiv 1$$

We wish to prove g.c.d. $(a, b) \equiv 1$

Suppose to the contrary that there is a polynomial $\theta(s)$ of degree ≥ 1 such that $\theta \mid a$ and $\theta \mid b$.

Then

$$x\,a + y\,b = x\,a_1\theta + y\,b_1\theta$$

$$= (x a_1 + y b_1)\theta$$

let $\lambda \in \mathbb{C}$ be such that $\theta(\lambda) = 0$

$$\Rightarrow \quad x(\lambda)a(\lambda) + y(\lambda)b(\lambda) = \left(x(\lambda)\,a_1(\lambda) + y(\lambda)b_1(\lambda)\right) \times \theta(\lambda)$$

$= 0$.

But $x(\lambda)a(\lambda) + y(\lambda)b(\lambda) = 1$ by hypothesis. Hence we have a contradiction. Hence $g.c.d.(a,b) \equiv 1$.

Applying Bezout's theorem to controller design.

Recall that given a rational, strictly proper transfer function

$$g(s) = \frac{b(s)}{a(s)}, \quad a, b \text{ coprime}$$

where $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$
and $b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m$
where $m \leq n-1$ and $b_0 \neq 0$, we can write,

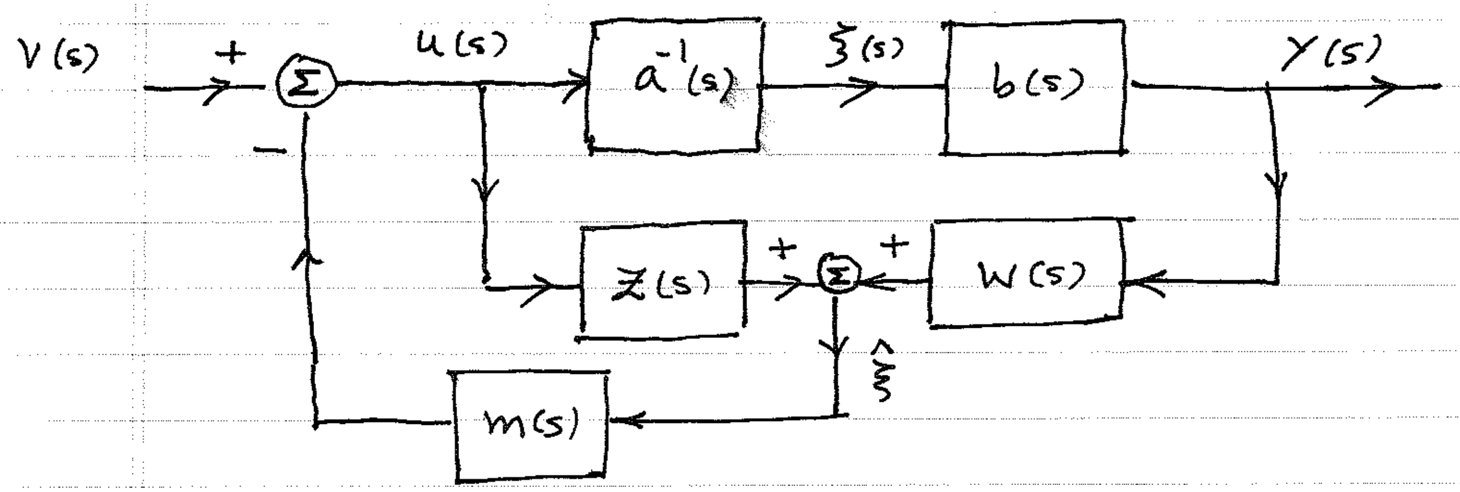
$$a(s)\,\xi(s) = u(s)$$

$$y(s) = b(s)\,\xi(s).$$

From coprimeness of a and b and Bezout, there exist unique polynomials $z(s)$ and $w(s)$, $\deg(z) < \deg(b)$, $\deg(w) < \deg(a)$ such that,

$$z a + w b \equiv 1 .$$

Consider the "controller" structure with $m(s)$ a polynomial:



Then
$$u(s) = V(s) - m(s)\,\hat{\xi}(s)$$
$$= V(s) - m(s)\,\big(z(s)\,u(s) + w(s)\,Y(s)\big)$$
$$= V(s) - m(s)\,\big(z(s)\,a(s)\,\xi(s) + w(s)\,b(s)\,\xi(s)\big)$$
$$= V(s) - m(s)\,\big(z(s)\,a(s) + w(s)\,b(s)\big)\,\xi(s)$$
$$= V(s) - m(s)\,\xi(s) \qquad \text{(Bezout)}$$

We thus note $\hat{\xi} = \xi$.

Hence

$$u(s) = a(s)\,\xi(s) = v(s) - m(s)\,\xi(s)$$

$$\Rightarrow \quad (a(s) + m(s))\,\xi(s) = v(s)$$

Thus the closed-loop transfer function is

$$g_{\text{closed}}(s) = \frac{b(s)}{a(s) + m(s)} = \frac{y(s)}{v(s)}.$$

The approach above has the serious flaw that, while $\frac{b(s)}{a(s)}$ is realizable as a finite dimensional linear system, the blocks $z(s)$, $w(s)$ and $m(s)$ are <u>not</u> realizable in the same sense since they are polynomials. Notice that the above structure is <u>reminiscent</u> of the observer - controller structure derived via state space theory. The situation can be remedied by using precisely this intuition:

Consider again state-space theory of

observer controller design

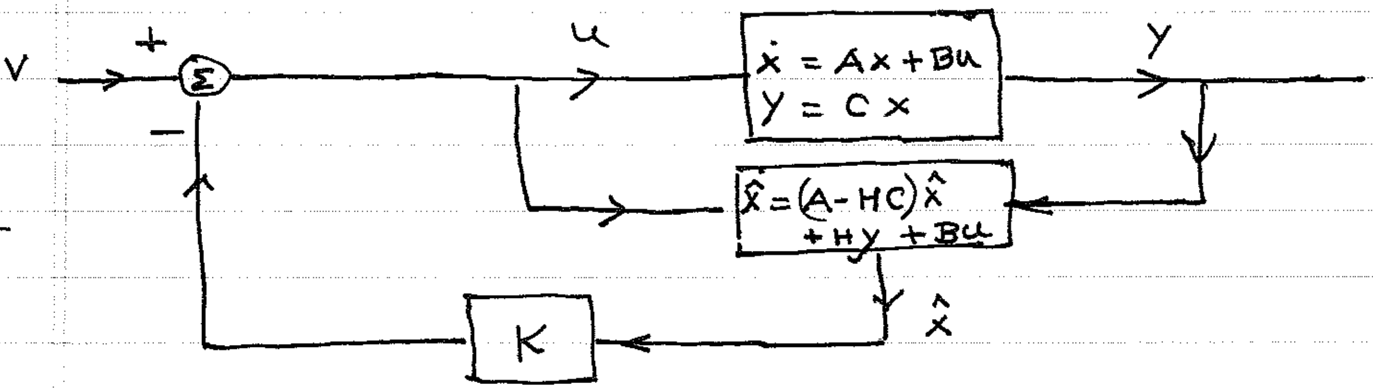


Fig 1

re-drawn as:

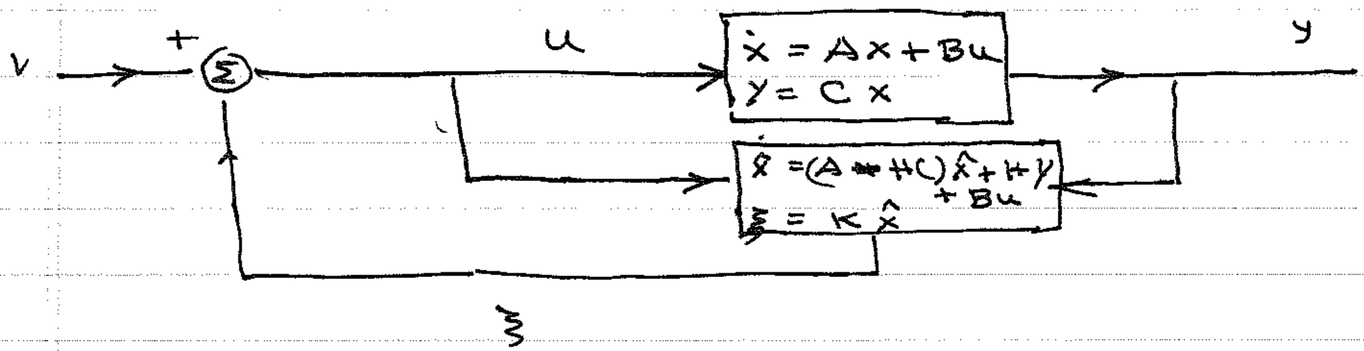


Fig2

and equivalently in frequency domain as:

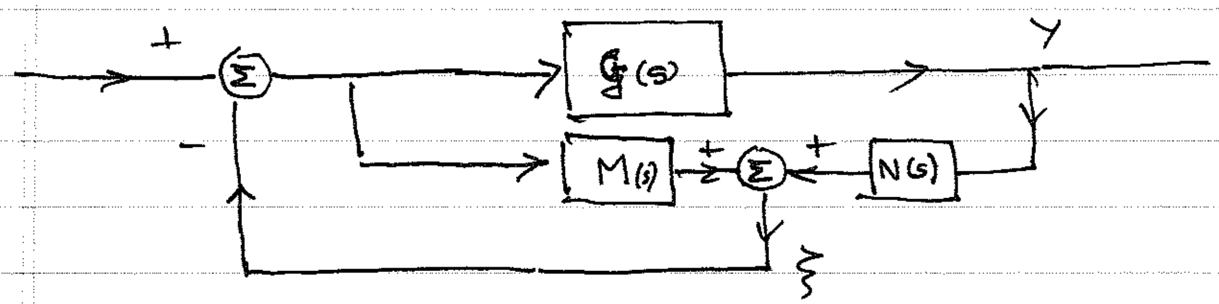


Fig 3