4. We study <u>properties</u> of the transition matrix associated to the differential equation

$$\frac{dx(t)}{dt} = A(t)\, x(t)$$

$$x(t_0) = x_0 .$$

The solution $x(t)$ is given by

$$x(t) = \Phi(t, t_0)\, x_0$$

where Φ is the transition matrix.

(i) <u>Abel - Jacobi - Liouville formula</u>:

$$\det(\Phi(t, t_0)) = \exp\left[\int_{t_0}^{t} tr(A(\sigma))\, d\sigma\right]$$

<u>proof</u>: Let c_{ij} denote the cofactor

(*) of the <u>element</u> ϕ_{ij} of the <u>matrix Φ</u>.

Then $$\det \Phi = \sum_{j=1}^{n} \phi_{ij}\, C_{ij}$$

Recall, from the definition of a cofactor, that ϕ_{ij} does not appear in C_{ij}. Hence

$$\frac{\partial}{\partial \phi_{ij}} \det \Phi = C_{ij}$$

By chain rule,

$$\frac{d}{dt}\det\left(\Phi(t,t_0)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial \det \Phi(t,t_0)}{\partial \phi_{ij}}\right]\frac{d}{dt}\phi_{ij}(t,t_0)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\ \frac{d}{dt}\phi_{ij}(t,t_0)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c^T_{ji}\ \frac{d}{dt}\phi_{ij}(t,t_0)$$

(where c^T_{ji} = $(j,i)^{th}$ element of the <u>transpose</u> of the matrix C of cofactors of Φ.)

$$= tr\left(C^T \frac{d}{dt}\Phi(t,t_0)\right)$$

$$= tr\left(C^T A\, \Phi\right)$$

$$= tr\left(A\, \Phi\, C^T\right)$$

$$= tr\left(A \det(\Phi)\cdot I\right) \qquad \text{(Lecture (1b) sec.13)}$$

$$= \det(\Phi)\ tr(A)$$

Integrating the above scalar differential equation, we obtain

$$\det(\Phi(t,t_0)) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(\sigma))\,d\sigma\right) \det(\Phi(t_0,t_0))$$

But $\det\left(\Phi(t_0,t_0)\right) = 1$.

Hence

$$\det\left(\Phi(t,t_0)\right) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(\sigma))\,d\sigma\right) \quad \blacksquare$$

(ii) $\Phi(t,t_0)$ is invertible.

proof: under the running hypothesis that elements of $A(t)$ are continuous functions of t on the interval $[t_0,t_1]$, it follows that $\int_{t_0}^{t} \operatorname{tr}(A(\sigma))\,d\sigma$ is a bounded function of t on the interval $[t_0,t_1]$ and hence its exponential is always positive. From (i) Φ is invertible. $\blacksquare$

(iii) $\Phi(t,t_0) = \Phi(t,t_1)\,\Phi(t_1,t_0)$

proof: For any x_0, $\Phi(t_1,t_0)\,x_0$ is the state $x_1 = x(t_1)$ attained by solving

$$\dot{x} = A x \quad ; \quad x(t_0) = x_0 .$$

Treating x_1 as initial state at time t_1

$x(t) = \Phi(t, t_1) x_1 = \Phi(t, t_1) \Phi(t_1, t_0) x_0$

is the state attained at time t for the same o.d.e.

On the other hand $\Phi(t, t_0) x_0$ is the state at time t for solving the same o.d.e starting at x_0 at t_0.

By uniqueness of solutions to linear time varying ordinary differential equations with continuous $A(t)$, it follows that

see pages 6-9 of Lecture 2(a)

$$x(t) = \Phi(t, t_0) x_0$$

$$= \Phi(t, t_1) \Phi(t_1, t_0) x_0 .$$

Since x_0 is arbitrary, the result follows ◪

Remark. No specific ordering of t, t_0 need be specified. The construction of section 3 of Lecture 2(a), (pages 9-16) makes sense whether $t \geq t_0$ or $t \leq t_0$.

Remark. Letting $t = t_0$, we see $\Phi(t_1, t_0)^{-1} = \Phi(t_0, t_1)$

Suppose we change basis in the state space $\mathbb{R}^n$ by a nonsingular transformation $P(t)$ which is time-dependent. In this ~~new~~ basis the state vector at each instant of time is represented by

$$z(t) = P(t)\, x(t).$$

Now, unless P is continuously differentiable w.r.t t, we cannot write a derivative for z. Assuming differentiability of P, we obtain

$$\begin{aligned} \dot{z} &= \dot{P} x + P \dot{x} \\ &= \dot{P} P^{-1} z + P A x \\ &= (\dot{P} P^{-1} + P A P^{-1}) z \end{aligned}$$

Let us denote by $\Phi_{\dot{P}P^{-1} + PAP^{-1}}(t, t_0)$ the transition matrix associated to the above differential equation for z. Then, for initial condition $z_0 = P(t_0) x_0$ the solution to the eqn for z is

$$z(t) = \Phi_{\dot{P}P^{-1} + PAP^{-1}}(t, t_0)\, z_0$$

$$= \Phi_{\dot{P}P^{-1}+PAP^{-1}}(t,t_0)\, P(t_0)\, x_0$$

The solution for the differential equation for x is given by

$$x(t) = \Phi_A(t,t_0)\, x_0$$

From the formula for $z(t)$ and from the change of basis we can also write

$$x(t) = P^{-1}(t)\, z(t)$$

$$= P^{-1}(t) \cdot \Phi_{\dot{P}P^{-1}+PAP^{-1}}(t,t_0) \cdot P(t_0)\, x_0 \qquad \forall x_0$$

By uniqueness of solutions (pages 6-9 of lecture 2(a)), we obtain equality of the above two formulas for $x(t)$, for all $x_0 \in \mathbb{R}^n$. It follows that the transition matrix for the system determining z is related to that of x by

$$\Phi_{\dot{P}P^{-1}+PAP^{-1}}(t,t_0) = P(t)\, \Phi_A(t,t_0)\, P^{-1}(t_0)$$

In the special case when $A(t) = A$ is a constant, ~~we~~ and $P(t) = P$ a constant, we see that

$$e^{(t-t_0)PAP^{-1}} = P\, e^{(t-t_0)A} P^{-1},$$

a result we already know by direct substitution of the similarity transformation in the power series expansion for the matrix exponential

$$\begin{aligned} e^{s\,PAP^{-1}} &= \sum_{k=0}^{\infty} \frac{(s\,PAP^{-1})^k}{k!} \qquad s \in \mathbb{R} \\ &= \sum_{k=0}^{\infty} \frac{P(sA)^k P^{-1}}{k!} \\ &= P\, e^{sA} P^{-1} \qquad \blacksquare \end{aligned}$$