Lecture 4(a) ENEE 660 Fall 2010

A consequence of the single-input control canonical form (Theorem, page 10 Lecture 3(b)) is that using a feedback law we can alter the characteristic polynomial to take a prescribed form.

Theorem (Eigenvalue / Pole placement for $m=1$)

Let $\dot{x} = Ax + bu$ be a controllable linear system. Then there exists a feedback $u = kx$ such that the characteristic polynomial

$$\chi_{A+bk}(s) = s^n + q_{n-1}s^{n-1} + \cdots + q_0$$

where q_i are prescribed.

Proof By controllability there exists P nonsingular such that

$$PAP^{-1} = A_c$$
$$Pb = b_c$$

where $[A_c, b_c]$ is in control canonical form (Theorem, page 10–15, Lecture 3(b)).

There exists $k_c = (-k^c_0, -k^c_1, \ldots, -k^c_{n-1})$

such that for $u = k_c Z = k_c P x$

$$A_c + b_c k_c$$

takes the form

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ -p_0 - k^c_0, & \ldots & & , & -p_{n-1} - k^c_{n-1} \end{bmatrix}$$

with characteristic polynomial

$$s^n + (p_{n-1} + k^c_{n-1}) s^{n-1} + \cdots + (p_0 + k^c_0).$$

Choose $k^c_i = q_i - p_i \qquad i = 0, 1, 2, \ldots, n-1$

Then $k = k_c P$

$P = T^{-1}$. see page 14 for T.

where P is as in the Theorem in Lecture 3(b) referred to earlier. $\square$

Remark A key practical application is to use feedback so that $A_c + b_c k_c$ has eigenvalues with negative real parts — asymptotic stability.

Remark There is an eigenvalue/pole placement theorem for the general case of systems with multiple inputs, i.e., $m > 1$. This basic result of LTI control theory can be derived by appealing to a result of Michael Heymann that reduces the problem to the single input case. The proof of Heymann's 1968 result can be given in a very insightful manner using an argument due to M.L.J. Hautus (1977). This argument uses the concept of invariant subspace.

Definition $A : X \to X$ is said to have an invariant subspace $V \subseteq X$ if $v \in V \Rightarrow Av \in V$. We usually write this as

$$AV \subseteq V$$

(clearly X is invariant; sometimes V is smaller than X)

Remark If $\lambda \in$ spectrum(A) and x is a corresponding nontrivial eigenvector then

$$V_\lambda = \{ v \in X : v = \alpha x, \ \alpha \in F \}$$

(field of scalars)

is an invariant subspace.

Lemma

Suppose $[A, B]$ is a controllable pair and $\mathcal{L}$ is subspace of the state space such that

$$A\mathcal{L} \subseteq \mathcal{L} \qquad \text{(invariance)}$$

(i.e. $x \in \mathcal{L} \Rightarrow Ax \in \mathcal{L}$),

and

$$\operatorname{im}(B) \subseteq \mathcal{L} \qquad \text{(containment)}$$

(i.e. $\forall\ u \in U \quad Bu \in \mathcal{L}$).

Then $\mathcal{L} = X$ the full state space.

Proof. By variation of constants formula for any t_1, and $x(0) = x_0$,

$$x(t_1) = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\sigma)} B\, u(\sigma)\, d\sigma.$$

By Cayley Hamilton theorem, we can write

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

for some specific continuous functions $\alpha_i(t)$.

For *any* continuous control functions $u(\cdot)$,

$$\int_0^{t_1} e^{(t-\sigma)A} B\, u(\sigma)\, d\sigma$$

$$= \sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k (t-\sigma)\, u(\sigma)\, d\sigma$$

Suppose $x_0 \in \mathcal{L}$. Since $A\mathcal{L} \subseteq \mathcal{L}$, it follows that $A^i x_0 \in \mathcal{L}$ and hence

$$e^{t_1 A} x_0 \in \mathcal{L}.$$

$B \int_0^{t_1} \alpha_k (t-\sigma)\, u(\sigma) d\sigma \in \mathcal{L}$ (since $\operatorname{im}(B) \subseteq \mathcal{L}$) and hence $A^k B \int_0^{t_1} \alpha_k (t-\sigma)\, u(\sigma)\, d\sigma \in \mathcal{L}$.
Thus,

$$\int_0^{t_1} e^{(t-\sigma)A} B\, u(\sigma)\, d\sigma \in \mathcal{L}$$

By linearity,

$$x(t_1) \in \mathcal{L}, \quad \text{for any } u(\cdot).$$

But, by controllability, $\exists$ $u(\cdot)$ such that *any* $x_1 = x(t_1)$. Hence $\mathcal{L} = X$. $\square$

Lemma (Hautus)

If $[A, B]$ is controllable and $b = Bv \neq 0$, then $\exists$ $u_1, u_2, \cdots, u_{n-1}$ such that

$$x_1 := b$$

$$x_{k+1} := A x_k + B u_k,$$

for $k = 1, 2, \ldots, n-1$, *defines* a set of linearly independent vectors $x_1, x_2, \cdots, x_n$ where n = dimension of state space X.

(Hautus 1977) **Proof**: $x_1 \neq 0$ and hence clearly linearly independent.

Suppose $x_1, x_2, \ldots, x_k$ are linearly independent. Let,

$$\mathcal{L} = \text{span}\{x_1, x_2, \ldots, x_k\}$$

$$= \left\{ x : \quad x = \sum_{i=1}^{k} \alpha_i x_i \quad \begin{array}{l} \alpha_i \in \mathbb{R} \\ i = 1, 2, \ldots, k \end{array} \right\}$$

Choose u_k such that

$$x_{k+1} = A x_k + B u_k \notin \mathcal{L}$$

If this is not possible, then

$$Ax_k + Bu \in \mathcal{L} \quad \forall u \in U$$

In particular, for $u = 0$

$$Ax_k \in \mathcal{L}.$$

Since $\mathcal{L}$ is a vector space,

$$Bu = Ax_k + Bu - Ax_k \in \mathcal{L} \quad \forall u \in U$$

Thus $\mathrm{im}(B) \subseteq \mathcal{L}$.

Also, for $i < k$

$$Ax_i = x_{i+1} - Bu_i \in \mathcal{L}$$

Thus $A\mathcal{L} \subseteq \mathcal{L}$.

From controllability of $[A, B]$, it follows that $\mathcal{L} = X$ the full state space X. Hence $\dim(\mathcal{L}) = \dim(X) = n$. ▨

Note: Sequence $u_1, u_2, \ldots, u_{n-1}$ is not unique.

Lemma (Heymann 1968)

If $[A, B]$ is controllable and $b = Bv \neq 0$, there exists a *linear feedback map* $F : X \to U$ such that

$$[A + BF \,,\, b] \quad \text{is controllable.}$$

Proof (by Hautus 1977)

Let *a* sequence $u_1, u_2, \ldots, u_{n-1}$ as in Hautus' lemma be given.
Define u_n *arbitrary*.
Define *the* linear map F by

$$F x_i = u_i \qquad i = 1, 2, \ldots, n.$$

Then,

$$(A + BF)^{k-1} b = x_k \qquad k = 1, 2, \ldots, n$$

To see this, note that this holds for $k = 1$ by definition. Suppose it holds for $k = l$. Then for $k = l + 1$

$$\begin{aligned}(A + BF)^{l+1-1} b &= (A + BF)^{l} b \\ &= (A + BF)(A + BF)^{l-1} b \\ &= (A + BF)\, x_l \qquad \text{(by hyp)}\end{aligned}$$

$$= A x_\ell + BFx_\ell$$

$$= A x_\ell + B u_\ell \qquad \text{(by definition of } F)$$

$$= x_{\ell+1} \qquad \text{(by Hautus' construction)}$$

We have the induction step.
It follows that the formula

$$(A+BF)^{k-1} b = x_k \qquad k = 1, 2, \cdots, n$$

holds. But $x_1, x_2, \cdots, x_n$ is a set of linearly independent vectors (by construction in Hautus' lemma).
Hence

$$[A+BF, b] \text{ is Controllable. } \square$$

Corollary Eigenvalue λ / Pole Placement Theorem.

Given $[A, B]$ controllable, there exists K such that $A+BK$ has a prescribed characteristic polynomial.

Proof Since $[A, B]$ is Controllable, there is a v such that $b = Bv \neq 0$.
From Heymann's Lemma, there exists

F such that $[A+BF, b]$ is controllable. It follows that, there is feedback (row vector) k such that

$$(A + BF + bk)$$

has prescribed spectrum (see Theorem 1-2 pages)

$$\begin{aligned} BF + bk &= BF + Bvk \\ &= B(F + vk) \end{aligned}$$

Define $K = F + vk$. ▨

Invariance of Controllability under Feedback

The property of controllability of a pair $[A, B]$ persists for $[A+BK, B]$ for any state feedback map K. To see this one could compute that the rank of

$$[B, (A+BK)B, \ldots, (A+BK)^{n-1}B]$$

is the same as the rank of

$$[B, AB, A^2B, \ldots, A^{n-1}B].$$

Such a calculation is not so insightful as verifying that

$$\dot{x} = (A+BK)x + Bv \qquad (*)$$

and

$$\dot{x} = Ax + Bu \qquad (**)$$

admit the same set of state trajectories.

Suppose, for a specific $\bar{u}(\cdot)$ in $(**)$ $\bar{x}(\cdot)$ satisfies

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$$

with $\bar{x}(t_0) = x_0$. Then if we define

$$v(t) = -K\bar{x}(t) + \bar{u}(t)$$

and substitute in $(*)$, we get

$$\dot{x}(t) = (A+BK)x(t) + B\bar{u}(t) - BK\bar{x}(t)$$

which admits $x(t) = \bar{x}(t)$ as a solution with $\bar{x}(t_0) = x(t_0) = x_0$. It is also the only solution to (*) with $x(t_0)=x_0$ and the chosen input $v(\cdot)$.

Even though we made the argument above for LTI systems, the result of invariance under state feedback holds in much greater generality, e.g. time-varying linear systems with linear continuous time-varying feedback. Thus the <u>reachability</u> / <u>controllability</u> properties of (linear) systems are invariant under state feedback.

<u>A house-keeping remark</u> [related to Lecture 3(b)].

In Lecture 3(b), page 6, we refer to

$$W(t_0, t_1) = L L^*$$

$$= \int_{t_0}^{t_1} \Phi(t_0, \sigma) B(\sigma) B^T(\sigma) \Phi^T(t_0, \sigma) d\sigma$$

as the reachability Gramian.

It is associated to solvability of the equation for $u(\cdot)$

$$x_0 - \Phi(t_0, t_1) x_1 = - \int_{t_0}^{t_1} \Phi(t_0, \sigma) B(\sigma) u(\sigma) d\sigma$$

We can associate two <u>special ~~eq~~ cases</u>:

(a) $x_0 = 0$.

Hence $$x_1 = \int_{t_0}^{t_1} \Phi(t_1, \sigma) B(\sigma) u(\sigma) d\sigma$$

Then the <u>reachable</u> subspace,

$$\mathcal{R}_{(0, t_1)} = \left\{ x_1 : \; x_1 = \int_{t_0}^{t_1} \Phi(t_1, \sigma) B(\sigma) u(\sigma) d\sigma \right\}$$

$$= \left\{ W_R(t_0, t_1) \eta \; : \; \eta \in \mathbb{R}^n \right\}$$

<u>where</u>

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \sigma) B(\sigma) B^T(\sigma) \Phi^T(t_1, \sigma) d\sigma$$

(b) $x_1 = 0$

Hence $$x_0 = - \int_{t_0}^{t_1} \Phi(t_0, \sigma) B(\sigma) u(\sigma) d\sigma$$

Then the subspace of initial states x_0 at time t_0, that can be transferred to 0 at time t_1

= <u>Controllable</u> subspace

$$= \mathcal{C}_{(0,t_1)} = \left\{ x_0 : x_0 = -\int_{t_0}^{t_1} \Phi(t_0,\sigma) B(\sigma) u(\sigma) d\sigma \right\}$$

$$= \left\{ W_c(t_0,t_1)\eta : \eta \in \mathbb{R}^n \right\}$$

where $$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\sigma) B(\sigma) B^T(\sigma) \Phi^T(t_0,\sigma) d\sigma$$

see Hespanha chapter 11 section 4 and further

It is <u>customary</u> to refer to $W_c(t_0,t_1)$ (same as the Gramian $W(t_0,t_1)$) as the <u>Controllability Gramian</u> (we called it the reachability Gramian). Our usage was dictated by studying a broader (beyond case (a) and (b)) question. Also, in much of the literature $W_R(t_0,t_1)$ is referred to as the <u>reachability</u> <u>Gramian</u>! Context should make things clear.