Lecture 4(c) ENEE 660 Fall 2010

Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$x(t_0) = x_0$$
$$y(t) = C(t)x(t)$$

where A is $n \times n$, B is $n \times m$, C is $p \times n$. Since knowledge of state $x(t)$ at each time t is useful in solving control problems, we ask:

Observability Problem: Is it possible, under suitable hypotheses, to reconstruct the state history $x(t)$ given the output history $y(t)$ for $t \in [t_0, t_1]$?

The idea here is to estimate/solve for the initial state x_0. If this can be done, without error, then using $u(\cdot)$, x_0 and the variation of constants formula one obtains the state history.

Given input $u(\cdot)$ and output $y(\cdot)$ over $[t_0, t_1]$, x_0 satisfies the linear equation,

$$\xi(t) \triangleq y(t) - \int_{t_0}^{t_1} C(t)\Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma$$

$$= C(t)\Phi(t,t_0)x_0 \qquad \forall\, t \in [t_0, t_1].$$

It follows that,

$$\int_{t_0}^{t_1} \Phi^T(t,t_0)\, C^T(t)\,\xi(t)\,dt$$

$$= \left(\int_{t_0}^{t_1} \Phi^T(t,t_0)\, C^T(t)\, C(t)\, \Phi(t,t_0)\,dt\right) x_0 .$$

Clearly, one can solve for x_0 <u>uniquely</u> <u>iff</u> the obs<u>ervability</u> <u>Gramian</u>

$$M(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(t,t_0)\, C^T(t)\, C(t)\, \Phi(t,t_0)\,dt$$

has only the vector 0 in its kernel, i.e. it has a <u>trivial kernel</u>. In that case <u>M</u> is <u>invertible</u> and one computes,

$$x_0 = (M(t_0,t_1))^{-1} \int_{t_0}^{t_1} \Phi^T(t,t_0)\, C^T(t)\,\xi(t)\,dt .$$

One then computes the state history by substituting for x_0 from above in the variation of constants formula.

Exercise :

If the input is known only upto additive (white Gaussian) noise vector $\eta(t)$ and the output is known only upto additive (white Gaussian) noise vector $\gamma(t)$, determine the uncertainty in x_0.

Suppose $A(t) \equiv A$ constant and $B(t) \equiv B$ constant, then the kernel of $M(t_0, t_1) = M(0, t_1 - t_0) = \int_0^{t_1 - t_0} e^{\sigma A^T} C^T C e^{\sigma A} d\sigma$ is the same as the kernel of $M_T = \sum_{k=0}^{n-1} (A^T)^k C^T C A^k$. If the kernel is trivial we say that the pair $[A, C]$ is observable.

The proof of this result is similar to that of the Theorem in page 6 of Lecture 3(b). Note that observability of

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \qquad (1)$$

is equivalent to controllability of

$$\dot{x} = -A^T x + C^T u \qquad (2)$$

To see this, note that the <u>reachability Gramian</u> associated to (2) is

$$W_{(2)}(t_0, t_1) = \int_{t_0}^{t_1} \Phi_{-A^T}(t_0, \sigma)\, C^T C\, \Phi_{-A^T}^T(t_0, \sigma)\, d\sigma$$

$$= \int_{t_0}^{t_1} e^{(\sigma - t_0)A^T} C^T C\, e^{(\sigma - t_0)A}\, d\sigma$$

$$= M_{(1)}(t_0, t_1),$$

the <u>observability Gramian</u> associated to (1).

For this reason, we ~~say~~ that systems (2) and (1) are duals of each other.

It is important to note that when the observability Gramian is invertible, the process of extracting state history from output history is apparently cumbersome requiring us to 'stop' at t_1, compute x_0 and then integrate the dynamics to obtain the state trajectory. It is desirable to have a method of extracting the state in dynamic or recursive manner. This can be done, if one can settle for an asymptotic result.

Theorem Let

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

be a linear time-invariant system with $[A, C]$ observable. Then there is a linear time-invariant system

$$\dot{\hat{x}} = F\hat{x} + Gu + Hy$$

such that $e(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$, for suitable choice of coefficient matrices F, G, H. (see proof for explicit choices).

Proof.

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - F\hat{x} - Gu - HCx \end{aligned}$$

$$= (A - HC)x - F\hat{x}$$

$$+ (B - G)u$$

Choosing $G = B$ and $F = A - HC$ we obtain the self-contained system

$$\dot{e} = (A - HC)e$$

Since $[A, C]$ is observable, $[A^T, C^T]$ is controllable. Hence, by the eigenvalue assignment theorem (Lecture 4(a), page 9, corollary) there exists a map T such that

$$\text{spectrum}\,(A^T - C^T T)$$

$$= \text{spectrum}\,((A - T^T C)^T)$$
$$= \text{spectrum}\,(A - T^T C)$$
$$\subseteq \mathbb{C}^- = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\}$$

Choose $H = T^T$. Then all eigenvalues of $\dot{e} = (A - HC)$ have negative real parts. Hence

$$e(t) \to 0 \quad \text{as} \quad t \to \infty. \quad \boxtimes$$

Remark The system (OBSERVER)

$$\dot{\hat{x}} = (A - HC)\hat{x} + Bu + Hy$$

with H such that spectrum $(A - HC) \subseteq \mathbb{C}^-$ is called an <u>asymptotic</u> (Luenberger) <u>observer</u> for the state of the original system (PLANT)

$$\dot{x} = Ax + Bu$$

$$y = Cx .$$

The observer is driven by both u and y. The state $\hat{x}(t)$ of the observer is an approximation to the state $x(t)$ of the PLANT. The approximation gets better and better as t increases, no matter what the initial error

$$e(0) = x(0) - \hat{x}(0)$$

is.

Example. Consider the system

$$\dot{x} = Ax + bu$$
$$y = cx$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -b \end{bmatrix} \qquad \begin{matrix} \omega > 0 \\ b > 0 \end{matrix}$$

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -b \end{bmatrix}$$ is invertible.

Therefore the system is observable.

For

$$H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$A - HC = \begin{bmatrix} 0 & 1-h_1 \\ -\omega^2 & -b-h_2 \end{bmatrix}$$

Choose $h_1 < 1$ and $h_2 > -b$ to ensure that

$$\chi_{A-HC}(s) = s^2 + s\,(b+h_2) + \omega^2(1-h_1)$$

has both roots in $\mathbb{C}^-$.

Then

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1-h_1 \\ -\omega^2 & -b-h_2 \end{bmatrix} \hat{x}$$

$$+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$+ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} y$$

is a Luenberger observer ~~for~~ the given plant.

The plant above is the linearization of the driven pendulum with friction about the straight down (stable) equilibrium. ω is the natural frequency and $b > 0$ is the coefficient of friction.

Example If we linearize the same plant about the straight up equilibrium then

$$A = \begin{bmatrix} 0 & 1 \\ \omega^2 & -b \end{bmatrix}$$

In this case $A - HC$ has the

Characteristic polynomial

$$\chi_{A-HC}(s) = s^2 + \lambda(b + h_2) + \omega^2(h_1 - 1)$$

Hence we must choose $h_2 > -b$ and $h_1 > 1$ to obtain a Luenberger observer.

<u>Use of an observer in feedback control</u>

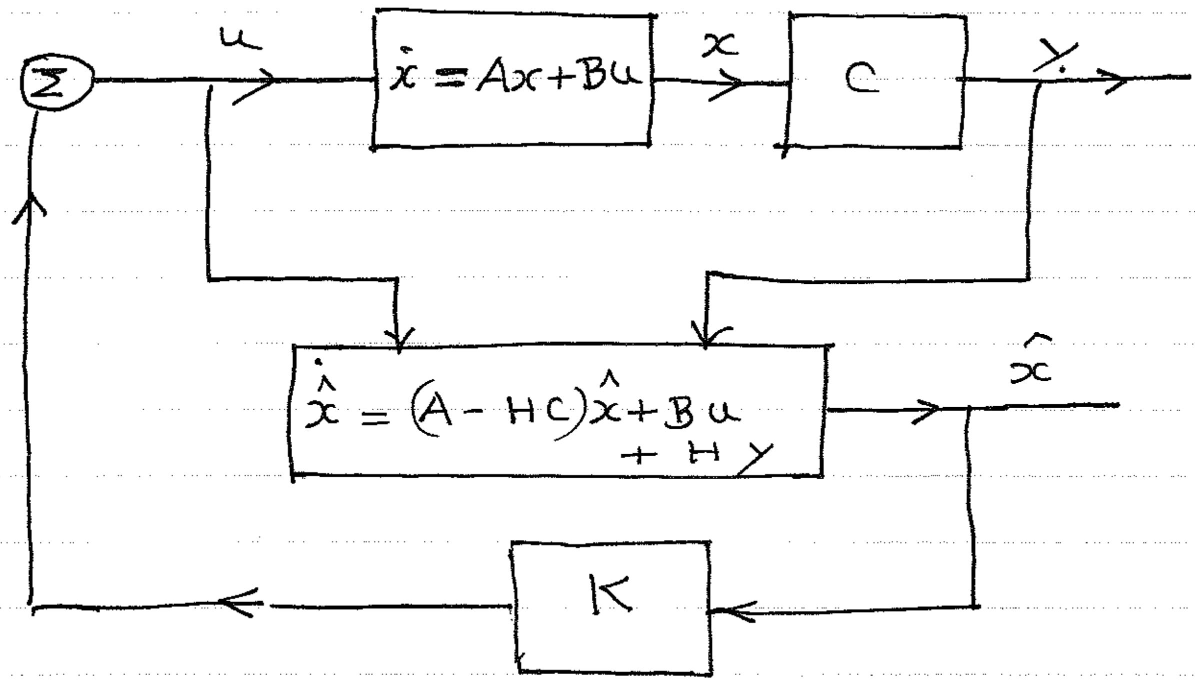
If
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is controllable and full state x is available (e.g. when $C = I$), then one can find a feedback control $u = Kx$ such that the <u>closed loop</u> system

$$\dot{x} = (A + BK)x$$

has spectrum $(A + BK) \subseteq \mathbb{C}^-$, thus stabilizing the 0 solution (equilibrium).

If state is not available but $[A, B]$ is controllable and $[A, C]$ is observable, then the following structure known as <u>Observer - controller structure</u> can be used.



where K is such that $(A + BK)$ has all eigenvalues in $\mathbb{C}^-$, and H is such that $(A - HC)$ has all eigenvalues in $\mathbb{C}^-$ and the control

$$u = K\hat{x}$$

(using state estimate $\hat{x}$ in place of x).

Since $\hat{x} = x - e$, the plant-observer-control dynamics takes the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax + BK\hat{x}\end{aligned}$$

$$= \quad Ax + BK(x-e)$$

$$= \quad (A \cancel{x} + BK)x - BKe$$

$$\dot{e} \quad = \quad (A - HC)e$$

Together we have the $2n$-dimensional block triangular system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The spectrum of this system is simply spectrum $(A+BK) \cup$ spectrum $(A-HC)$

$$\subseteq \mathbb{C}^-$$

Hence $x(t) \to 0$, $e(t) \to 0$.
So the origin is stabilized.

The problem of selecting H is neatly decoupled from that of selecting K we see a separation principle at work here : if observability holds, estimate the state and use the estimate in place of the true state in the controller.