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A little detour on measure etc

< see Real Analysis by H. Royden for full scoop >

(a) Any set Ω has associated to it the set 2^{Ω} of all subsets of Ω. Consider a subcollection $\mathcal{A} \subset 2^{\Omega}$ of subsets of Ω satisfying

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c$ (the complement of A in Ω) belongs to $\mathcal{A}$

(iii) $A_n \in \mathcal{A},\ n = 1, 2, \ldots \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

Such a collection $\mathcal{A}$ is called a <u>σ-algebra</u> of subsets of Ω. Clearly 2^{Ω} itself is a σ-algebra. So is the (very small) collection $\{\phi, \Omega\}$ where ϕ denotes the empty set.

(b) A <u>measurable space</u> is a pair $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of subsets of Ω.

(c) A <u>set function</u> $\mu : \mathcal{A} \longrightarrow [0, \infty) \cup \{+\infty\}$ is a <u>measure</u> if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for $A_1, A_2, A_3, \ldots$ a sequence in $\mathcal{A}$ such that $A_i \cap A_j = \phi$ for $i \neq j$

If further $\mu(\Omega) < \infty$, then we say that it is a finite measure.

It is clear that $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

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(d) suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then

$f : \Omega_1 \to \Omega_2$ is said to be a <u>measurable mapping</u>, provided $B \in \mathcal{A}_2 \Rightarrow f^{-1}(B) \in \mathcal{A}_1$.

(e) Given any $\mathcal{Q} \subset 2^{\Omega}$, there is a σ-algebra $\mathcal{A}$ such that $\mathcal{Q} \subset \mathcal{A} \subset 2^{\Omega}$ and furthermore it is the <u>smallest</u> σ-algebra with this property. We refer to it as the σ-algebra generated by $\mathcal{Q}$, often denoted as $\mathcal{A} = \sigma[\mathcal{Q}]$.

(f) suppose $\Omega = \mathbb{R}^1$ and $\mathcal{Q}$ = collection of all open intervals of $\mathbb{R}^1$. Then the σ-algebra generated by $\mathcal{Q}$ is referred to as the <u>Borel σ-algebra $\mathcal{B}(\mathbb{R}^1)$</u>. It can be shown that every interval, open, closed or semi-open, in $\mathbb{R}^1$ is in $\mathcal{B}(\mathbb{R}^1)$ and is thus a <u>Borel subset</u> of $\mathbb{R}^1$.

(g) We say that $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a <u>measurable function</u> if

$$f^{-1}(B_2) \in \mathcal{B}(\mathbb{R}^1)$$

whenever $B_2 \in \mathcal{B}(\mathbb{R}^1)$

(h) Any continuous function is automatically a measurable function.

(i) There is a unique measure, the Lebesgue measure $\mu : \mathcal{B}(\mathbb{R}^1) \to [0, \infty) \cup \{+\infty\}$ such that $\mu([a,b)) = b - a = \mu([a,b]) = \mu((a,b])$.

Lebesgue measure is clearly <u>not</u> a finite measure.

(j) A function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is said to be simple if it is of the form

$$f(x) = \sum_{i=1}^{N-1} c_i \, \chi_{[x_i,\, x_{i+1})}(x)$$

where $-\infty < x_1 < x_2 < \cdots < x_N < \infty$, N finite, $c_i \in \mathbb{R}^1$ and

$$\chi_{[a,b)}(x) = \begin{cases} 1 & a \le x < b \\ 0 & \text{otherwise} \end{cases}$$

(k) The integral $I(f) = \int_{-\infty}^{\infty} f(x) \, \mu(dx)$ where $\mu(\cdot)$ is the Lebesgue measure can be defined <u>for simple functions</u> as

$$I(f) = \sum_{i=1}^{N-1} c_i \, (x_{i+1} - x_i)$$

<u>Theorem</u> If $\{f_n\}$ is a sequence of simple functions,

$$f_n(x) = \sum_{i=1}^{N(n)-1} c_i^n \, \chi_{[x_i^{(n)},\, x_{i+1}^{(n)})}(x)$$

and similarly

$$g_n(x) = \sum_{i=1}^{N(n)-1} d_i^n \, \chi_{[y_i^{(n)},\, y_{i+1}^{(n)})}(x)$$

and $\quad f_n \le f_{n+1}, \quad g_n \le g_{n+1} \qquad n = 1, 2, \ldots$

and $\quad f_n \to f$ as $n \to \infty$ and $g_n \to g$, as $n \to \infty$ then

$$\lim_{n \to \infty} I(f_n) = \lim_{n \to \infty} I(g_n)$$ ▨

Using the above result and using the fact that for each measurable function f there exists $\{f_n\}$ a sequence of simple functions, $f_n \le f_{n+1}$ and $f_n \to f$ as $n \to \infty$, we can define unambiguously

$$I(f) = \lim_{n \to \infty} I(f_n)$$

for any measurable function. This is the <u>Lebesgue integral</u> of f.

(e) Two functions $f_i : [0, \infty) \to \mathbb{R} \quad i = 1, 2$ are said to be equivalent $f_1 \sim f_2$ if

$$\mu(\{x : f_1(x) \ne f_2(x)\}) = 0.$$

We denote the equivalence class of f to be $[f]$.

The spaces we define below are all spaces of <u>equivalence classes</u> of measurable

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functions. ~~To~~ avoid awkward ~~notions~~ notations, we will continue to use f when we really mean $[f]$.

(m) The L_p spaces: $1 \le p < \infty$

$$L_p[0,\infty) = \{ f : [0,\infty) \to \mathbb{R} \mid f \text{ measurable}, \int_0^\infty |f(x)|^p \mu(dx) < \infty \}$$

$$L_\infty[0,\infty) = \{ f : [0,\infty) \to \mathbb{R} \mid f \text{ measurable}, \operatorname*{ess\,sup}_{[0,\infty)} |f(x)| < \infty \}$$

< here $\operatorname*{ess\,sup}_{[0,\infty)} f = \sup_{[0,\infty)-A} f$ where A is a suitable set of measure zero >

<u>Theorem</u>

(i) L_p with $\|f\|_p = \left(\int_0^\infty |f(x)|^p \mu(dx) \right)^{1/p}$ is complete.

(ii) L_∞ with $\|f\|_\infty = \operatorname*{ess\,sup}_{[0,\infty)} |f(x)|$ is complete

(iii) L_2 is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^\infty f(t)\, g(t)\, \mu(dt)$$

and $\|f\|_2 = \sqrt{\langle f, f \rangle}$