Control, Observation and Feedback: a State Space View

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Systems as Physical Objects

- Have inputs, states and outputs

- Can be interconnected

- Distinguish as plants, controllers, filters, reference signal generators, data converters, etc

- Wide variety of technological, biological, logistical and economic examples fit this perspective

Systems as Mathematical Objects

– A family of transformations (depending on input signals) of states

– A family of read-out maps (depending on input signals)

Causal models derived from ordinary and partial differential equations.

$$\dot{x}(t) = f(x(t),\ u(t))$$

$$y(t) = h(x(t),\ u(t))$$

$x(t) \in X$ state space
$u(t) \in U$ input space
$y(t) \in Y$ output space

Systems as Mathematical Objects

Structure in models

$$\begin{aligned} \dot{x}(t) &= f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) \\ y(t) &= h(x(t)) \end{aligned}$$

Here $f_i,\ i = 0, 1, 2, ..., m$ are vector fields on state space

Linearity:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Stationarity: A, B, C, D time invariant.

Descriptions of Systems (Internal)

Parameters: A, B, C, D

Input-state response:

$$x(t) = \Phi_A(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t, \sigma)B(\sigma)u(\sigma)d\sigma,$$

where $\Phi_A(\cdot, \cdot)$, the transition matrix, satisfies

$$\dot{\Phi}_A(t, t_0) = A(t)\Phi_A(t, t_0)$$

and $\Phi_A(t_0, t_0) = \mathbb{1}$, the identity matrix.

Easy to verify

$$\begin{aligned} \Phi_A(t, t_0) &= \mathbb{1} + \int_{t_0}^{t} A(\sigma)d\sigma \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{\sigma_1} A(\sigma_1)A(\sigma_2)d\sigma_2 d\sigma_1 + \ldots \end{aligned}$$

Descriptions of Systems (External)

Input-Output response:

$$
\begin{aligned}
y(t) &= C(t)\Phi_A(t,t_0)x(t_0) \\
&\quad + \int_{t_0}^{t} C(t)\Phi_A(t,\sigma)B(\sigma)u(\sigma)d\sigma \\
&\quad + D(t)u(t) \\
&= y_0(t) + \int_{t_0}^{t} W(t,\sigma)u(\sigma)d\sigma \\
&\quad + D(t)u(t)
\end{aligned}
$$

Weighting pattern:

$$
W(t,\sigma) = C(t)\Phi_A(t,\sigma)B(\sigma).
$$

$$
\text{Drift} = y_0(t)
$$

depends only on initial conditions.

Specializing to Time-invariant Setting

$$\begin{aligned}\Phi_A(t,t_0) &= e^{A\cdot(t-t_0)} \\ W(t,t_0) &= Ce^{A\cdot(t-t_0)}B\end{aligned}$$

Impulse response

$$= Ce^{At}B + D\delta(t)$$

Transfer function

$$G(s) = C(s\mathbb{1} - A)^{-1}B + D$$

Descriptions of Systems (Internal vs External)

Change of variables

$$z(t) = P(t)x(t)$$

changes the internal description but not the external one.

In the time-invariant setting, two internal descriptions with parameters $[A, B, C, D]$ and $[PAP^{-1}, PB, CP^{-1}, D]$ have the same transfer function.

Transfer functions are proper, and if $D = 0$, strictly proper (i.e., $G(s) \to 0$ as $s \to \infty$)

Reachability Problem

Given $x_0 =$ state at time t_0, does there exist a control $u(\cdot)$ defined on the time interval $[t_0, t_1]$ that drives the system to x_1 at time t_1?

<u>Define</u> $R^{t_1}_{(x_0,t_0)}$ set of such x_1.

$$R_{(x_0,t_0)} = \bigcup_{t_1>0} R^{t_1}_{(x_0,t_0)}$$

We say that the system is reachable from (x_0, t_0) if $R_{(x_0,t_0)} =$ state space

Reachability Problem

For linear systems with $X = \mathbb{R}^n, U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ solve

$$\begin{aligned}(x_0 - \Phi_A(t_0, t_1)x_1) &= -\int_{t_0}^{t_1} \Phi_A(t_0, \sigma)B(\sigma)u(\sigma)d\sigma \\ &= L(u)\end{aligned}$$

System is reachable from (x_0, t_0) if $\mathcal{R}(L) =$ range space of $L = \mathbb{R}^n$

Define reachability gramian $W(t_0, t_1) = LL^*$, where

$$\begin{aligned}L^* : \mathbb{R}^n &\rightarrow C^m[t_0, t_1] \\ \eta &\mapsto -B'(\cdot)\Phi'_A(t_0, \cdot)\eta,\end{aligned}$$

where $'$ denotes transpose of a matrix.
Then $\mathcal{R}(L) = \mathcal{R}(W)$

Reachability Problem

Suppose there exists $\eta \in \mathbb{R}^n$ such that

$$(x_0 - \Phi_A(t_0, t_1)x_1) = W(t_0, t_1)\eta.$$

Then, control defined by

$$u_0(t) = -B'(t)\Phi'_A(t_0, t)\eta$$

drives the system from (x_0, t_0) to (x_1, t_1).

System is reachable iff W is invertible.

If u is any other control that drives (x_0, t_0) to (x_1, t_1), then

$$\int_{t_0}^{t_1} u'(\sigma)u(\sigma)d\sigma \geqslant \int_{t_0}^{t_1} u'_0(\sigma)u_0(\sigma)d\sigma$$

Observability Problem

Is it possible to determine the initial state $x(t_0)$ from an input-output pair known over a time interval $[t_0, t_1]$? If yes, we say the system is observable. For linear systems, define the drift map

$$\begin{aligned} P : \mathbb{R}^n &\rightarrow C^p[t_0, t_1] \\ x_0 &\mapsto C(\cdot)\Phi_A(\cdot, t_0)x_0 \end{aligned}$$

Define observability gramian

$$\begin{aligned} M(t_0, t_1) &= P^*P \\ &= \int_{t_0}^{t_1} \Phi'_A(\sigma, t_0)C'(\sigma)C(\sigma)\Phi_A(\sigma, t_0)d\sigma \end{aligned}$$

Since null space $\mathcal{N}(P) = \mathcal{N}(M(t_o, t_1))$, it follows that the system is observable iff $M(t_0, t_1)$ is invertible.

Gramians and the Time-invariant Setting

For time-invariant linear systems

$W(t_0, t_1)$ is invertible for $t_1 > t_0$

$\leftrightarrow [B, AB, ...A^{n-1}B]$ has rank n
$\leftrightarrow [s\mathbb{1} - A|B]$ has constant rank n
for all $s \in \mathbb{C}$

$M(t_0, t_1)$ is invertible for $t_1 > t_0$

$$\leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has rank } n$$

$$\leftrightarrow \left[\frac{C}{s\mathbb{1} - A}\right] \text{ has constant rank } n$$

for all $s \in \mathbb{C}$

Realization Problem

We have already seen that internal representations are not uniquely defined by an external representation – the change of variables idea.

It can be shown that any weighting pattern $W(t,\sigma)$ that is factorizable in the form

$$W(t,\sigma) = Q(t)R(\sigma)$$

where $Q(t)$ is $p \times n_1$ and $R(\sigma)$ is $n_1 \times m$, admits a finite dimensional representation

$$W(t,\sigma) = C(t)\Phi_A(t,\sigma)B(\sigma)$$

Realization Problem

The finite dimensional representation (realization) above, is of the lowest possible (state space) dimension $= n$, iff it is both reachable and observable.

All minimal state space realizations are related by the (possibly time-dependent) change of variables formula.

In the time-invariant setting realizability is equivalent to the condition that the transfer function $G(s)$ is a $p \times m$ matrix of strictly proper, rational functions.

Realization Problem

For a time-invariant linear system, let the transfer function be given by a Laurent series

$$G(s) = \sum_{i=0}^{\infty} \frac{L_i}{s^{i+1}}.$$

L_i are called Markov parameters.
Clearly $G(s)$ is strictly proper. It is rational iff the infinite Hankel matrix

$$\begin{bmatrix} L_0 & L_1 & L_2 & \cdots \\ L_1 & L_2 & L_3 & \cdots \\ L_2 & L_3 & L_4 & \cdots \\ \vdots & & & \end{bmatrix}$$

is of finite rank $= n$, called the McMillan degree.
A classic realization algorithm (Ho-Kalman), constructs a minimal realization from Markov parameter data.

Minimality, Poles and Zeros

For $m = p = 1$ a strictly proper

$$
\begin{aligned}
G(s) &= \frac{q(s)}{p(s)} \\
&= \frac{q_{n-1}s^{n-1} + ... + q_0}{s^n + p_{n-1}s^{n-1} + ... + p_0}
\end{aligned}
$$

$$
\begin{aligned}
q(s) \text{ and } p(s) & \quad \text{relatively prime,} \\
\text{Poles } \{G(s)\} &= \text{roots of } p(s) \\
\text{Zeros } \{G(s)\} &= \text{roots of } q(s)
\end{aligned}
$$

If $G(s) = C(s\mathbb{1} - A)^{-1}B$, then
Poles $\{G(s)\} \subset$ spectrum (A).

Minimality, Poles and Zeros, cont'd

Poles $\{G(s)\}$ = spectrum (A)
iff $[A, B, C]$ is minimal.

In that case

$$\begin{aligned}\text{McMillan degree} &= \text{degree of } p(s) \\ &= \text{dimension of state space}\end{aligned}$$

Remark on System Identification

The realization problem (passing from the sequence $\{L_i\}_{i=1}^{\infty}$ to a minimal triple $[A, B, C]$) was viewed as an idealized form of the identification problem.

In practice, pre-processing of data from input-output experiments into the sequence $\{L_i\}_{i=0}^{\infty}$ is not the preferred way. There are alternatives founded on statistical methodologies, e.g., the canonical correlation analysis dues to Akaike (1977).

Remark on System Identification

System identification algorithms may be viewed as dynamical systems on spaces of transfer functions. For $m = p = 1$, the space $Rat(n)$ of strictly proper rational functions of McMillan degree n was identified as an object of study by Brockett(1976). $Rat(n)$ has very interesting topological and geometric structures. It admits interesting dynamics - e.g., a flow equivalent to the famous integrable system of Toda.

Gramians and Reduction

For time-invariant linear systems $[A, B, C]$, with spectrum $(A) \subseteq \mathbb{C}^-$, let

$$W_c = \int_0^\infty exp\ (tA)BB'\, exp\ (tA')dt$$

and

$$W_o = \int_0^\infty exp\ (tA')C'C exp\ (tA)dt$$

A minimal triple $[A, B, C]$ is balanced iff $W_c = W_o = \sum = diag\,(\sigma_1, \sigma_2, ..., \sigma_n)$ where the Hankel singular values

$$\sigma_i = (\lambda_i(W_c W_o))^{1/2} \qquad\qquad i = 1, 2, ..., n$$

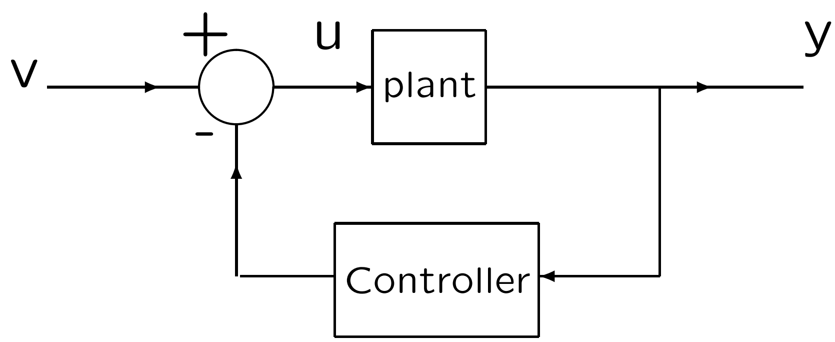
Gramians and Reduction, cont'd

and are ordered such that

$$\sigma_i \geq \sigma_{i+1}$$

Balancing + truncation $\rightarrow$ reduction.
Connections to PCA.

Closing the Loop



$$[A, B, C,] \;\rightarrow\; [A - BKC,\; B,\; C]$$

$$\begin{array}{rcl} \dot{x} &=& Ax + Bu \\ y &=& Cx \end{array} \;\rightarrow\; \begin{array}{rcl} \dot{x} &=& (A - BKC)x + Bv \\ y &=& Cx \end{array}$$

$$W(t, \sigma) \;\rightarrow\; W^f(t, \sigma)$$

where

$$W^f(t, t_0) = W(t, t_0) - \int_{t_0}^{t} W(t, \sigma) K(\sigma) W^f(\sigma, t_0) d\sigma$$

Closing the Loop, cont'd

In the time-invariant case

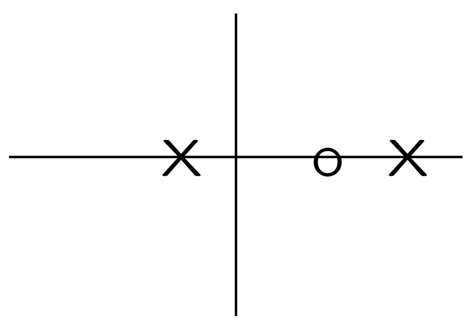
$$\begin{aligned} G^f(s) &= G(s) - G(s)KG^f(s) \\ \leftrightarrow G^f &= (\mathbb{1} + GK)^{-1}G \end{aligned}$$

Closing the Loop

Feedback alters the system response. But there are severe limits to how much can be done with a constant gain controller.

Example:

$m = p = 1$
$g(s)$ with pole-zero pattern



Output feedback cannot move the r.h.p. pole to the left.

Root-locus calculations tell us why.
Dynamic compensators help.

State Feedback

Consider the linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Under state feedback

$$u = -Kx + v$$

$$[A, B, C] \to [A - BK, B, C].$$

State feedback preserves reachability properties (set of attainable state trajectories is unchanged), but not observability properties.

State Feedback

In the single input setting, there is a change of variables such that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot\cdot & 0 \\ 0 & \cdot & & & \\ 0 & \cdot & . & . & 1 \\ -p_0 & -p_1 & \cdot\cdot & \cdot\cdot & -p_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ : \\ 1 \end{bmatrix}$$

the control canonical form. In this case letting $\chi_A(s)$ denote the characteristic polynomial of A, there is $K = (k_0, k_1, ..., k_{n-1}) s.t.$

$$\begin{aligned} \chi_{A-BL}(s) &= s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_0 \\ &= \beta(s) \end{aligned}$$

for <u>any</u> polynomial β.

State Feedback

Essentially, the same idea can be used to show

Theorem (Pole Placement)
Let $[A, B]$ be a controllable pair. Then there exists a state feedback K $s.t.$

$$\chi_{A-BK}(s) = \beta(s)$$

for any desired polynomial $\beta(s)$ of degree $n =$ dimension of state space.

Remark There is a famous canonical form associated with P. Brunovsky for controllable pairs $[A, B]$, under the feedback group

$$\begin{array}{lll} A \to & PAP^{-1} & B \to PB \\ A \to & A - BK & B \to B \\ A \to & A & B \to BQ \end{array}$$

State Feedback

One application of the pole placement theorem is to find a feedback law $u = -Kx + v$ such that all eigenvalues of $(A - BK)$ are in $\mathbb{C}^-$ the open $l.h.p.$ The theorem guarantees such a K. There are a number of approaches to finding such stabilizing feedback laws. See lectures of Khaneja.

A Difficulty with State Feedback

- State variables are typically not directly measurable.

- A set of linear combinations of state variables may be all that is available.

- Is there a way to stabilize the system?

Idea: Use output variables to estimate the state (asymptotically). Then substitute estimates for actual state.

Observer/Estimator

Consider

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

a time invariant system.

Assume that $[A, B, C]$ is minimal. Consider the associated system

$$\dot{\hat{x}} = (A - \Gamma C)\hat{x} + Bu + \Gamma y$$

(This new system accepts as inputs, the original inputs u together with the outputs of the original system.)

Observer/Estimator, cont'd

Let $e = x - \widehat{x}$.
Then $\dot{e} = (A - \Gamma C)e$.

Observability of $[A, C]$
$\leftrightarrow$ Reachability of $[A', C']$
$\leftrightarrow$ spectrum assignability of $(A' - C'\Gamma')$
$\leftrightarrow$ spectrum assignability of $(A - \Gamma C)$
(by pole placement theorem)

Observer/Estimator

Choose Γ so that all eigenvalues of $(A - \Gamma C)$ are in $\mathbb{C}^-$.
Then, as $t \to \infty$,

$$\begin{aligned} e(t) &\to 0 \\ x(t) &\to \widehat{x}(t) \end{aligned}$$

Thus the state $\widehat{x}(t)$ asymptotically estimates $x(t)$.

Suppose K is such that $(A - BK)$ has spectrum $\subset \mathbb{C}^-$. Consider the closed loop system

$$\dot{x} = Ax + B(-K\widehat{x} + v)$$

obtained by using the state estimate $\widehat{x}$ in place of x.

Observer/Estimator/Controller

Then the combinaton of observer, plant and controller takes the form

$$
\begin{aligned}
\dot{x} &= Ax - BK(x-e) + Bv \\
&= (A - BK)x + BKe + Bv \\
\dot{e} &= (A - \Gamma C)e
\end{aligned}
$$

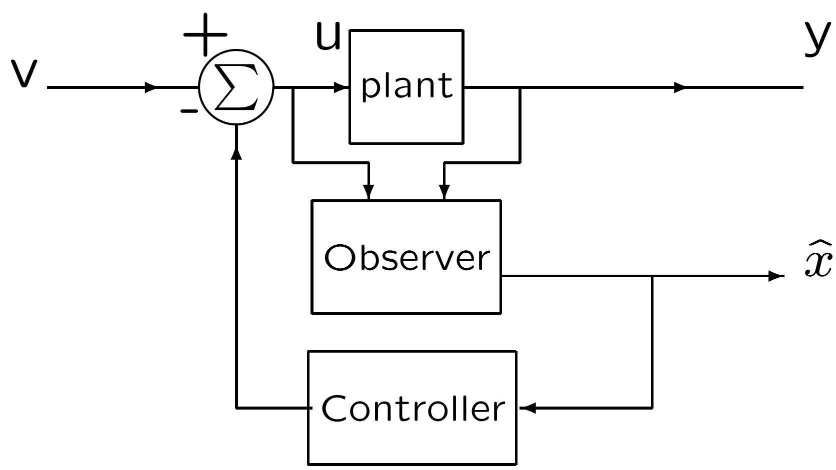
The *overall* dynamics is governed by the matrix

$$A_{overall} = \begin{bmatrix} A - BK & BK \\ 0 & A - \Gamma C \end{bmatrix}$$

choice of K and Γ as above ensures that spectrum $(A_{overall}) \subseteq \mathbb{C}^-$

Separation Theorem

The control system structure above takes the form



This is a powerful prototype for control system design in which the choices of controller and observer parameters can be made in a separated/independent manner.

Separation Theorem

We return to this idea in later lectures of Khaneja and James from a variety of perspectives.