Queen Dido's problem. (part of Lecture 7)

Find a curve in the $x-y$ plane such that it and the horizontal axis bound the maximum area for a given ~~perimeter~~ lengths of the curve.

We let 't' denote the curve parameter.

The curve $\gamma : [0,T] \to \mathbb{R}^2$

$$t \mapsto (x(t), y(t))$$

has velocity function $\frac{d\gamma}{dt}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

and the arc-length function

$$s(t) = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Assume that curve is regular i.e. the speed

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \neq 0$$

for any t. Then $s(t)$ is strictly monotone increasing in t and can be inverted to express t as a function of s. We again denote $\gamma = \gamma(t(s))$ as the same curve parametrised by the arc-length parameter s, $\gamma : [0, \ell] \to \mathbb{R}^2$ where,

$$\ell = \text{total length of the curve}$$
$$= \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^T \frac{ds}{dt}\, dt$$
$$= \int_0^\ell ds \qquad \text{as it should be}$$

The speed of the curve in the arc length parametrization is

$$\left\| \frac{d\gamma}{ds} \right\| = \left\| \frac{d\gamma}{dt} \frac{dt}{ds} \right\|$$
$$= \left\| \frac{d\gamma}{dt} \right\| \cdot \left| \frac{dt}{ds} \right|$$
$$= \frac{1}{\left(\frac{ds}{dt}\right)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$= \frac{ds/dt}{ds/dt}$$
$$= 1$$

So we call this the <u>unit speed</u> parametrization as well.

So we now state ~~the~~ the problem of Dido in terms of the arc length parametrized curve $s \mapsto \gamma(s)$:

Find $\gamma(s)$ such that it bounds maxm area with the horizontal axis and

$$\gamma(0) = (0,0)$$

$$\gamma(\ell) = (x(\ell),\ 0)$$

(The $x(\ell)$ coordinate is unspecified),

We now use the overdot $\dot{}$ to denote differentiation w.r.t. s. Then

area $= \displaystyle\int_0^\ell y(s)\,\frac{dx}{ds}\,ds$

$= \displaystyle\int_0^\ell y\,\dot{x}\,ds$

Note that from the unit speed property,

$$\dot{x}^2 + \dot{y}^2 = 1 \quad \Rightarrow \quad \dot{x} = \sqrt{1-(\dot{y})^2}$$

So we have a calculus of variations problem with fixed end-points.

$$\boxed{\begin{array}{l} \text{Maximize} \quad \displaystyle\int_0^\ell y(s)\,\sqrt{1-(\dot{y}(s))^2}\,ds \\ \text{subject to} \quad y(0)=0\,; \quad y(\ell)=0\,. \end{array}}$$

The space of functions $s \mapsto y(s)$ over which it makes sense, is the closed set

of functions continuously differentiable in s and
$|\dot{y}(s)| \leq 1$.

$\boxed{\text{Necessary Conditions}}$.

Euler Lagrange.

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

where

$$L = L(s, y(s), \dot{y}(s))$$

$$= y(s)\sqrt{1-(\dot{y}(s))^2}$$

$$\frac{\partial L}{\partial y} = \sqrt{1-\dot{y}^2}$$

$$\frac{\partial L}{\partial \dot{y}} = -\frac{y\dot{y}}{\sqrt{1-\dot{y}^2}}$$

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{y}} = \frac{-\dot{y}\dot{y} - y\ddot{y}}{\sqrt{1-\dot{y}^2}} - \frac{y\dot{y}\,(-\tfrac{1}{2})(-2\dot{y})(\ddot{y})}{(1-\dot{y}^2)^{3/2}}$$

$$= -\frac{(\dot{y}\dot{y}+y\ddot{y})(1-\dot{y}^2) + y\dot{y}^2\ddot{y}}{(1-\dot{y}^2)^{3/2}}$$

$$= -\frac{\dot{y}^2 + y\ddot{y} - \dot{y}^4 - y\dot{y}^2\ddot{y} + y\dot{y}^2\ddot{y}}{(1-\dot{y}^2)^{3/2}}$$

$$= -\frac{\dot{y}^2(1-\dot{y}^2) + y\ddot{y}}{(1-\dot{y}^2)^{3/2}}$$

$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y}$$

$$= -\frac{\dot{y}^2(1-\dot{y}^2)+y\ddot{y}}{(1-\dot{y}^2)^{3/2}} - (1-\dot{y}^2)^{1/2}$$

$$= \frac{-\dot{y}^2+\dot{y}^4-y\ddot{y} - (1-\dot{y}^2)^2}{(1-\dot{y}^2)^{3/2}}$$

$$= \frac{-\dot{y}^2+\dot{y}^4-y\ddot{y}-1+2\dot{y}^2-\dot{y}^4}{(1-\dot{y}^2)^{3/2}}$$

$$\Rightarrow \boxed{y\ddot{y}-\dot{y}^2+1 = 0}$$

End-point conditions $y(0)=0=y(\ell)$. Verify that for $y(s) = \frac{\ell}{\pi}\sin\left(\frac{\pi s}{\ell}\right)$

$$y\ddot{y}-\dot{y}^2+1 = \frac{\ell}{\pi}\sin\left(\frac{\pi s}{\ell}\right)\left(-\frac{\pi}{\ell}\sin\left(\frac{\pi s}{\ell}\right)\right) - \left(\cos\left(\frac{\pi s}{\ell}\right)\right)^2+1$$

$$= 0$$

How do we get $x(s)$?

$$\dot{x}(s) = \sqrt{1-(\dot{y}(s))^2}$$

$$= \sqrt{1-\left(\cos\left(\frac{\pi s}{\ell}\right)\right)^2}$$

$$= \sin\left(\frac{\pi s}{\ell}\right)$$

$$x(s) = \int_0^s \dot{x}(\sigma)\,d\sigma = \int_0^s \sin\left(\frac{\pi\sigma}{\ell}\right)d\sigma$$

$$= \frac{\ell}{\pi}\left(-\cos\left(\frac{\pi s}{\ell}\right)\right)\Big|_0^s = \frac{\ell}{\pi}\left(1 - \cos\left(\frac{\pi s}{\ell}\right)\right)$$

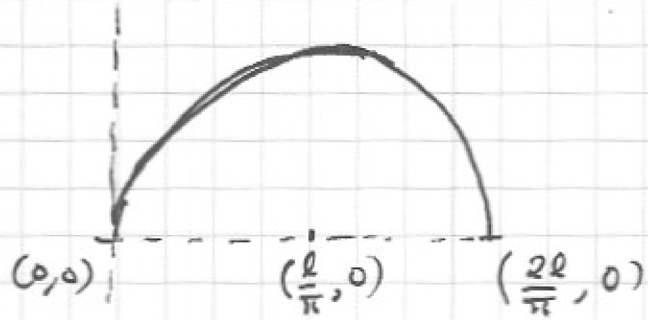
Thus the curve $\gamma(s) = \left(\left(1 - \cos\left(\frac{\pi s}{\ell}\right)\right)\frac{\ell}{\pi}, \frac{\ell}{\pi}\sin\frac{\pi s}{\ell}\right)$

$$\left(x(s) - \frac{\ell}{\pi}\right)^2 + (y(s))^2$$

$$= \cos\left(\frac{\pi s}{\ell}\right)^2 \frac{\ell^2}{\pi^2} + \frac{\ell^2}{\pi^2}\sin^2\left(\frac{\pi s}{\ell}\right)$$

$$= \frac{\ell^2}{\pi^2}$$

a semi-circular arc of radius $\frac{\ell}{\pi}$, center $\left(\frac{\ell}{\pi}, 0\right)$



How do we know this is the only solution to E-L?

In the next page we approach this question via the use of a conserved quantity. This approach is useful for a wide class of problems in the calculus of variations.

Conserved Quantities (for general time-independent Lagrangian)

For Lagrangians that do not depend on t explicitly, there is a conserved quantity applicable to extremals:

$$\boxed{E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L.}$$

To verify this, compute

$$\frac{dE}{dt} = \frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right)$$

$$= \dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \ddot{x}\frac{\partial L}{\partial \dot{x}} - \frac{dL}{dt}$$

$$= \dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \cancel{\ddot{x}\frac{\partial L}{\partial \dot{x}}} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x}\dot{x} - \cancel{\frac{\partial L}{\partial \dot{x}}\ddot{x}}$$

$$= \dot{x}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) - \frac{\partial L}{\partial t}$$

$$= 0 \qquad \left(\text{from E-L and time independence of } L\right)$$

Thus E is constant for any solutions to E-L.

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apply this to Dido's problem. $L = L(y,\dot{y}) = y\sqrt{1-\dot{y}^2}$

$$E = \dot{y}\frac{\partial L}{\partial \dot{y}} - L$$

$$= \frac{-y\dot{y}\dot{y}}{\sqrt{1-\dot{y}^2}} - y\sqrt{1-\dot{y}^2}$$

$$= \text{constant} = C$$

$$\Rightarrow \quad \frac{-y\dot{y}^2 - y(1-\dot{y}^2)}{\sqrt{(1-\dot{y}^2)}} = c$$

$$\Rightarrow \quad \boxed{y^2 + c^2\dot{y}^2 = c^2}$$

We solve this equation.

$$\frac{dy}{\sqrt{c^2-y^2}} = \frac{ds}{|c|}$$

$$\int_0^y \frac{dy}{\sqrt{c^2-y^2}} = \int_0^s \frac{ds}{|c|}$$

Case (i) $\quad c \neq 0$

Use $y = c\sin(\theta)$ a change of variables, on the left, to get,

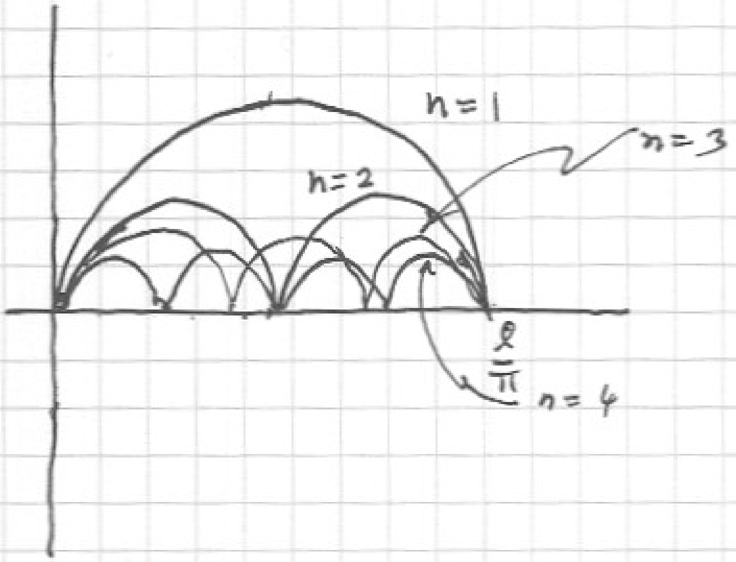
$$\boxed{\int_0^{\sin^{-1}(y/c)} \frac{c\cos\theta\, d\theta}{|c|\cos\theta} = \frac{s}{|c|}}$$

$$\Rightarrow \quad \boxed{y = c\sin\left(\frac{s}{c}\right)}$$

Boundary Conditions $\quad y(0) = y(\ell) = 0$

$$\sin\left(\frac{\ell}{c}\right) = 0 \qquad \frac{\ell}{c} = \pm n\pi \iff c = \frac{\ell}{n\pi} \quad n \in \mathbb{Z}$$

This gives us (broken) extremals as in the figure below:



as $n \to \infty$ the solution goes to the zero solution with zero area (minimum). It is clear that $n=1$ is <u>the</u> maximum.

<u>Figure</u> Extremal Solutions to Dido's Problem

<u>Remark</u>: Study the paper of P.D. Lax (American Mathematical Monthly (1995), vol <u>102</u>, No. 2, February issue, pp 158–159) and compare this to the approach based on the Calculus of variations.

(a link to this paper is provided on the course web page.)