Aim of lecture on Wednesday, March 12, 2014

To outline steps in the proof of the Lagrange multiplier theorem.

Definition Let X be a normed linear space and $g_i : X \to \mathbb{R}$, $i = 1, 2, 3, \ldots, n$ be Fréchet differentiable functionals on X. We say that $x_0 \in X$ is a regular point of the constraint set

$$\Omega = \{ x \in X : g_i(x) = 0, \ i = 1, 2, \ldots, n \}$$

if $x_0 \in \Omega$ (i.e. $g_i(x_0) = 0$ $i = 1, 2, \ldots, n$), and the Fréchet derivatives

$$f_i = (D g_i)_{x_0} \qquad i = 1, 2, \ldots, n$$

are linearly independent functionals.

[note: this necessarily requires $n \leq \dim X$]

Implicit Function Theorem

Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ have a Fréchet derivative $(Df)_{(\theta, x)}$ for each $(\theta, x) \in S$ an open subset of $\mathbb{R}^m \times \mathbb{R}^n$, and $(Df)_{(\theta, x)}$ is continuous in (θ, x).

Let (θ_0, x_0) be such that

$f(\theta_0, x_0) = 0$. and the partial Fréchet derivative $(D_2 f)_{(\theta_0, x_0)}$ is nonsingular.

Then there exist open neighborhoods, U of $\theta_0 \in \mathbb{R}^m$ and V of $x_0 \in \mathbb{R}^n$ such that, for each $\theta \in U$, the equation

$$f(\theta, x) = 0$$

has a unique solution $x \in V$. Moreover the solution can be given as

$$x = g(\theta)$$

so that $f(\theta, g(\theta)) = 0 \quad \theta \in U$, with g Fréchet differentiable near θ_0 and $(Dg)_\theta$ is continuous in θ at $\theta = \theta_0$.

Remark This theorem can be extended to Banach spaces.

Remark In the proof of Theorem 3 (leading up to the Lagrange multiplier theorem), we use a case with $m=1$, and n = the number of constraint functionals defining Ω.

Theorem 1 (Lecture 5(b), page 3).

- Let X be a vector space. Let $f_i : X \to \mathbb{R}$, $i = 1, 2, \cdots, n$ be a set of linearly independent linear functionals. Then there exist vectors $x_1, x_2, \cdots, x_n \in X$ such that

the matrix $[f_i(x_j)]$ is nonsingular.

Remark Here, necessarily, $n \leq \dim(X)$. The proof of Theorem 1 is an argument by mathematical induction.

The vectors $x_1, x_2, \ldots, x_n$ obtained in Theorem 1 are linearly independent, see Lecture 5(b), page 5.

Theorem 2 (Lecture 5(b), page 6)

Let $f, f_1, f_2, \cdots, f_n$ be linear functionals on X. Suppose, for each $x \in X$ such that $f_i(x) = 0$, $i = 1, 2, \cdots, n$, we have $f(x) = 0$

[In short hand $\bigcap_{i=1}^{n} \mathrm{Ker}(f_i) \subset \mathrm{Ker}(f)$.]

Then, $f = \sum_{i=1}^{n} \alpha_i f_i$ for some $\alpha_i \in \mathbb{R}$.

Remark In Lecture 5(b) pages 6–7, a proof is given assuming that the functionals $f_1, f_2, \cdots, f_n$ are linearly independent. There is no loss of generality in doing this.

More seriously, the proof requires $\dim(X) = m < \infty$. In that case $\dim(X^*) = m$ and $m \geq n$.

Developing a proof for X, possibly infinite dimensional, is <u>left as an exercise</u>, with a hint.

Theorem 3 Consider a functional $g : X \to \mathbb{R}$. Let $\Omega = \{x \in X : g_i(x) = 0,\ i = 1, 2, \ldots, n\}$ be a <u>constraint set</u> defined by the functionals g_i, $i = 1, 2, \ldots, n$.

If x_0 is an <u>extremum</u> of g subject to the constraints <u>and</u> if x_0 is a <u>regular point</u> of Ω, then

$$\bigcap_{i=1}^{n} \operatorname{Ker}\left((Dg_i)_{x_0}\right) \subset \operatorname{Ker}\left((Dg)_{x_0}\right).$$

Remark. A proof is given in lecture 5(c) pages 1–3.

A cleaned up and more explanatory proof write-up in a set of <u>Updated Version</u> notes (also at the class web-site) is available.

<u>Theorem</u> [Lagrange multiplier]

If $x_0 \in \Omega$ is an extremum of $g : X \to \mathbb{R}$ subject to constraint Ω <u>and</u> if x_0 is a regular point of Ω, then there exists scalars λ_i, $i = 1, 2, \ldots, n$ such that

$$\left(D\left(g + \sum_{i=1}^{n} \lambda_i g_i\right)\right)_{x_0} = 0 .$$

We call λ_i the Lagrange multipliers. ☒

<u>Remark</u> Proof follows very directly from Theorem 3 and Theorem 2.