# NOTES AND CORRECTIONS: "STOLARSKY'S INVARIANCE PRINCIPLE FOR FINITE METRIC SPACES" [1]

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*[Please see also the latest update of the arXiv preprint 2005.12995 where these notes have been incorporated]*

*1. Correction.* Theorem 6.1 in [1] requires a correction: the signs in Eq.(72) should be flipped.

THEOREM 6.1: Let $\mathcal{X}$ be a finite metric space, let $Z \subset \mathcal{X}$ be a subset of size $N$, and let $g_i \ge 0, i = 0, 1, \dots, n$ and $\gamma(t) := \sum_{i=t}^n g_i$. Then

$$D_G^{L_2}(Z) = \langle \lambda_G \rangle_{\mathcal{X}} - \langle \lambda_G \rangle_Z, \tag{72}$$

where for $x, y \in \mathcal{X}$

$$\lambda_G(x,y) := \frac{1}{2} \sum_{z \in \mathcal{X}} |\gamma(d(x,z)) - \gamma(d(y,z))|.$$

In the proof, Eq. (74) and (75) should to be corrected as follows:

$$\begin{aligned}
\sum_{t=0}^n g_t |B(x,t) \cap B(y,t)| &= \sum_{t=0}^n g_t \sum_{z \in \mathcal{X}} \mathbb{1}_{B(x,t)}(z) \mathbb{1}_{B(y,t)}(z) = \sum_{z \in \mathcal{X}} \sum_{t=0}^n g_t \mathbb{1}_{B(x,t)}(z) \mathbb{1}_{B(y,t)}(z) \\
&= \sum_{z \in \mathcal{X}} \sum_{t=\max(d(z,x),d(z,y))}^n g_t = \sum_{z \in \mathcal{X}} \gamma(\max(d(z,x))(d(z,y))) \\
&= \sum_{z \in \mathcal{X}} \min(\gamma(d(z,x)), \gamma(d(z,y))) \\
&= \sum_{z \in \mathcal{X}} \frac{1}{2} \{\gamma(d(z,x)) + \gamma(d(z,y)) - |\gamma(d(z,x)) - \gamma(d(z,y))|\}
\end{aligned} \tag{74}$$

$$\sum_{t=0}^n g_t |B(u,t)|^2 = \frac{1}{|\mathcal{X}|} \sum_{x,y \in \mathcal{X}} \sum_{z \in \mathcal{X}} \frac{1}{2} \{\gamma(d(x,z)) + \gamma(d(y,z)) - |\gamma(d(x,z)) - \gamma(d(y,z))|\}. \tag{75}$$

Accordingly, the first displayed line on p.183 should take the form

$$D_G^{L_2}(Z) = \langle \lambda_G \rangle_{\mathcal{X}} - \frac{1}{N} \sum_{w=1}^n A_w \lambda_G(w).$$

*2. Simple proof of (37).* The expression for the sum of squares of the Krawtchouk polynomials (37) affords a simple proof, given here. In [1] this result is referenced to [2] where the proof is substantially more complicated. We will prove that

$$\sum_{k=0}^n (K_k^{(n)}(i))^2 = (-1)^i K_n^{(2n)}(2i) = \binom{2n-2i}{n-i} \binom{2i}{i} \Big/ \binom{n}{i}. \tag{37}$$

Start with the generating function of the Krawtchouk polynomials:

$$\sum_{k=0}^{n} K_k^{(n)}(x) z^k = (1+z)^{n-x}(1-z)^x. \tag{33}$$

Then

$$\begin{aligned}\sum_{k=0}^{n} K_k^{(n)}(w) z^k \sum_{k'=0}^{n} K_{k'}^{(n)}(w) z^{-k'} &= (1+z)^{n-w}(1-z)^w (1+z^{-1})^{n-w}(1-z^{-1})^w \\ &= (-1)^w z^{-n}(1+z)^{2(n-w)}(1-z)^{2w} \\ &= (-1)^w z^{-n} \sum_{k=0}^{2n} K_k^{(2n)}(2w) z^k.\end{aligned}$$

Equating the constant terms on both sides, we obtain the first equality in (37),

$$\sum_{k=0}^{n} (K_k^{(n)}(i))^2 = (-1)^i K_n^{(2n)}(2i).$$

Next, again using (33),

$$\sum_{k=0}^{2n} K_k^{(2n)}(n) z^k = (1-z^2)^n,$$

and thus, $K_{2k}^{(2n)}(n) = (-1)^k \binom{n}{k}$. Now (34) implies that $\binom{2n}{n} K_{2k}^{(2n)}(n) = \binom{2n}{2k} K_n^{(2n)}(2k)$, and we get

$$K_{2k}^{(2n)}(n) = \frac{\binom{2n}{2k} K_n^{(2n)}(2k)}{\binom{2n}{n}} = (-1)^k \binom{n}{k}$$

or

$$K_n^{(2n)}(2i) = (-1)^i \frac{\binom{n}{i}\binom{2n}{n}}{\binom{2n}{2i}}.$$

Rewriting the binomial coefficients on the right, we obtain the second equality in (37).

Apparently, (37) has no nonbinary analog, i.e., there is no closed-form expression for the sum of squares of Krawtchouk polynomials in general. The same seems true for the Eberlein polynomials, so calculations of discrepancy in the Johnson space also do not look immediate.

## References

[1] A. Barg, Stolarsky's invariance principle for finite metric spaces, *Mathematika*, 67 (2021), 158–186.

[2] P. Feinsilver and R. Fitzgerald, The spectrum of symmetric Krawtchoul matrices, *Linear Alg. Appl.*, 235 (1996), pp.121–139.

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